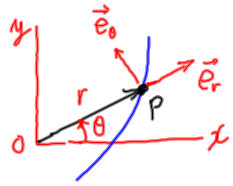
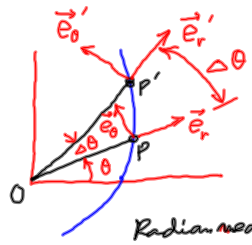
Acceleration of a particle in polar coordinates



Position vector of particle P is

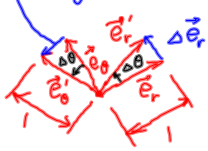
$$\boxed{\vec{r} = r\,\vec{e}_r} \quad \text{V.V.} \cdots \text{I.}$$

$\vec{e}_r$ & $\vec{e}_\theta$ are unit vectors, but not constant vectors.

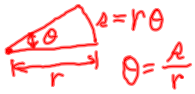


$$\Delta\vec{e}_\theta = 1\cdot\Delta\theta(-\vec{e}_r) = -\Delta\theta\,\vec{e}_r$$

$$\Delta\vec{e}_r = 1\cdot\Delta\theta\,\vec{e}_\theta = \Delta\theta\,\vec{e}_\theta$$



Radian measure formula



$$\frac{d\vec{e}_r}{dt} = \lim_{\Delta t\to 0}\frac{\Delta\vec{e}_r}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t}\vec{e}_\theta = \frac{d\theta}{dt}\vec{e}_\theta = \dot{\theta}\,\vec{e}_\theta$$

$$\boxed{\dot{\vec{e}}_r = \dot{\theta}\,\vec{e}_\theta} \qquad \boxed{\dot{\vec{e}}_\theta = -\dot{\theta}\,\vec{e}_r} \quad \text{V.V.} \cdots \text{I.}$$

$$\frac{d\vec{e}_\theta}{dt} = \lim_{\Delta t\to 0}\frac{\Delta\vec{e}_\theta}{\Delta t} = -\lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t}\vec{e}_r = -\frac{d\theta}{dt}\vec{e}_r = -\dot{\theta}\,\vec{e}_r$$

Velocity = time rate of change of position vector.

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}, \quad \vec{r} = r\,\vec{e}_r, \quad \vec{v} = \dot{\vec{r}}$$

$$\vec{v} = \dot{r}\,\vec{e}_r + r\,\dot{\vec{e}}_r = \dot{r}\,\vec{e}_r + r(\dot{\theta}\,\vec{e}_\theta)$$

$$\boxed{\vec{v} = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta} \qquad v_r = \dot{r}, \quad v_\theta = r\dot{\theta}$$

Acceleration = time rate of change of velocity

$$\vec{a} = \dot{\vec{v}} = \ddot{r}\,\vec{e}_r + \dot{r}\,\dot{\vec{e}}_r + \dot{r}\dot{\theta}\,\vec{e}_\theta + r\ddot{\theta}\,\vec{e}_\theta + r\dot{\theta}\,\dot{\vec{e}}_\theta$$

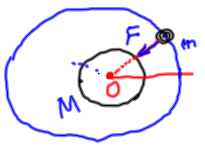
$$= \ddot{r}\,\vec{e}_r + \dot{r}(\dot{\theta}\,\vec{e}_\theta) + \dot{r}\dot{\theta}\,\vec{e}_\theta + r\ddot{\theta}\,\vec{e}_\theta + r\dot{\theta}(-\dot{\theta}\,\vec{e}_r)$$

$$\boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta}$$

$$a_r = \ddot{r} - r\dot{\theta}^2 \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

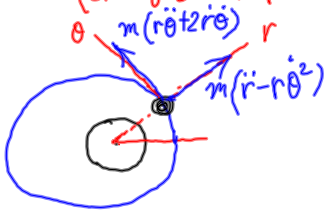
Central-force motion (Free flight of spacecraft)

(method of force & acceleration)



Free-body diagram

=



Effective-force diagram (mass × acceleration)

$+\nearrow \Sigma F_r$: $\boxed{-F = m(\ddot{r} - r\dot{\theta}^2)}$

$+\nwarrow \Sigma F_\theta$: $\boxed{0 = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})}$

$$F = \frac{GMm}{r^2} = GMm\,r^{-2}$$