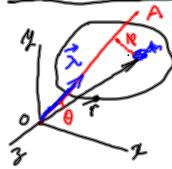


Mass moments of inertia (See Fig 858)



$$I_{xx} = \int r_x^2 dm = \int (y^2 + z^2) dm$$

$$I_{yy} = \int r_y^2 dm = \int (z^2 + x^2) dm$$

$$I_{zz} = \int r_z^2 dm = \int (x^2 + y^2) dm$$

$$I_{xy} = \int xy dm \quad I_{yz} = \int yz dm \quad I_{zx} = \int zx dm$$

$$I_{yx} = I_{xy} \quad I_{zy} = I_{yz} \quad I_{xz} = I_{zx}$$

$$I_{OA} = \int r^2 dm \quad \vec{r} = r \sin \theta = r (\lambda_x \hat{x} + \lambda_y \hat{y} + \lambda_z \hat{z})$$

$$\vec{r}^2 = (\lambda_x \hat{x} + \lambda_y \hat{y} + \lambda_z \hat{z}) \cdot (\lambda_x \hat{x} + \lambda_y \hat{y} + \lambda_z \hat{z})$$

$$\vec{r}^2 = (3\lambda_z - y\lambda_y)^2 + (x\lambda_y - 3\lambda_x)^2 + (y\lambda_x - x\lambda_y)^2$$

$$\begin{aligned} \vec{r}^2 &= (3\lambda_z - y\lambda_y)^2 + (x\lambda_y - 3\lambda_x)^2 + (y\lambda_x - x\lambda_y)^2 \\ &= (3^2 \lambda_z^2 - 2 \cdot 3y \lambda_y \lambda_z + y^2 \lambda_y^2) + (x^2 \lambda_y^2 - 2 \cdot 3x \lambda_x \lambda_y + 3^2 \lambda_x^2) \\ &\quad + (y^2 \lambda_x^2 - 2yx \lambda_x \lambda_y + x^2 \lambda_y^2) \end{aligned}$$

$$\begin{aligned} I_{OA} &= \int \lambda_z^2 (y^2 + z^2) dm + \int \lambda_y^2 (3^2 + x^2) dm + \int \lambda_x^2 (x^2 + y^2) dm \\ &\quad - 2 \int \lambda_x \lambda_y x y dm - 2 \int \lambda_y \lambda_z y z dm - 2 \int \lambda_z \lambda_x x z dm \end{aligned}$$

$$I_{OA} = I_{xx} \lambda_z^2 + I_{yy} \lambda_y^2 + I_{zz} \lambda_x^2 - 2 I_{xy} \lambda_x \lambda_y - 2 I_{yz} \lambda_y \lambda_z - 2 I_{zx} \lambda_z \lambda_x$$

Parallel-axis theorem

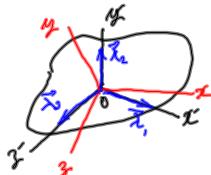
$$I_{xx} = \bar{I}_{xx} + m(\bar{z}^2 + \bar{y}^2) \quad I_{yy} = \bar{I}_{yy} + m(\bar{z}^2 + \bar{x}^2)$$

$$I_{zz} = \bar{I}_{zz} + m(\bar{x}^2 + \bar{y}^2) \quad I_{xy} = \bar{I}_{xy} + m\bar{x}\bar{y}$$

$$I_{yz} = \bar{I}_{yz} + m\bar{y}\bar{z} \quad I_{zx} = \bar{I}_{zx} + m\bar{z}\bar{x}$$

(Study Example 19.7)

Rotation of coordinate axes



From Oxyz to O'x'y'z', we define a rotation matrix

$$R = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}$$

$$\text{Transpose of } R: \quad R^T = \begin{bmatrix} \lambda_{11} & \lambda_{21} & \lambda_{31} \\ \lambda_{12} & \lambda_{22} & \lambda_{32} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{bmatrix}$$

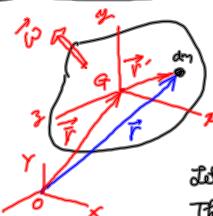
Transformation law for inertia matrices

(MEEG 4703)

$$I' = R I R^T$$

Momentum of a rigid body

Let GXYZ be embedded in the rigid body at its mass center G.



$$\vec{r} = \vec{r}' + \vec{r}''$$

$$\vec{r} = \vec{r}' + (\frac{d\vec{r}}{dt})_{GXYZ}$$

Let $\vec{\omega}$ be the angular velocity of the body.

$$\text{Then, } \vec{\omega} = \vec{\omega} = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k}$$

$$\vec{v} = \vec{v}' + \vec{\omega} \times \vec{r}' \quad \vec{v} = \vec{v}' + \vec{\omega} \times \vec{r}'$$

Let \vec{L} be the linear momentum vector of the body, and \vec{H}_G be the angular momentum vector of the body about G.

$$\begin{aligned} \vec{L} &= \int \vec{r} dm = \int (\vec{r}' + \vec{\omega} \times \vec{r}') dm = \int \vec{r}' dm + \int \vec{\omega} \times \vec{r}' dm \\ &= \vec{r}' \int dm + \vec{\omega} \times \int \vec{r}' dm = \vec{r}' m + \vec{\omega} \times (m \vec{r}') = m \vec{v}' \end{aligned}$$

$$\boxed{\vec{L} = m \vec{v}'}$$

\vec{r}' = vector drawn from G to G

$$\vec{H}_G = \int \vec{r}' \times \vec{r} dm = \int \vec{r}' \times (\vec{v}' + \vec{\omega} \times \vec{r}') dm$$

$$= (\int \vec{r}' dm) \times \vec{v}' + \int \vec{r}' \times (\vec{\omega} \times \vec{r}') dm$$

$$= \vec{r}' m + \int \vec{r}' \times (\vec{\omega} \times \vec{r}') dm = \int \vec{r}' \times (\vec{\omega} \times \vec{r}') dm$$

$$\boxed{\vec{H}_G = \int \vec{r}' \times (\vec{\omega} \times \vec{r}') dm}$$