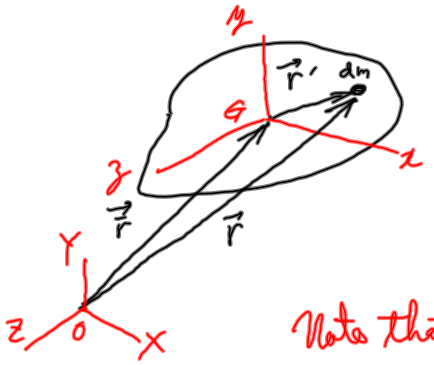
Kinetic energy of a rigid body



$$\vec{v} = \vec{\bar{v}} + \vec{\omega} \times \vec{r}' \qquad T = \frac{1}{2}\int v^2\, dm$$

$$v^2 = \vec{v}\cdot\vec{v} = (\vec{\bar{v}} + \vec{\omega}\times\vec{r}')\cdot(\vec{\bar{v}} + \vec{\omega}\times\vec{r}')$$

$$= \bar{v}^2 + \vec{\bar{v}}\cdot(\vec{\omega}\times\vec{r}') + (\vec{\omega}\times\vec{r}')\cdot\vec{\bar{v}} + (\vec{\omega}\times\vec{r}')\cdot(\vec{\omega}\times\vec{r}')$$

Note that $\vec{A}\cdot\vec{P}\times\vec{Q} = \vec{A}\times\vec{P}\cdot\vec{Q}$

$$v^2 = \bar{v}^2 + 2\vec{\bar{v}}\cdot(\vec{\omega}\times\vec{r}') + \vec{\omega}\times\vec{r}'\cdot(\vec{\omega}\times\vec{r}')$$

$$T = \frac{1}{2}\int \bar{v}^2\, dm + \vec{\bar{v}}\cdot\vec{\omega}\times\cancel{\int \vec{r}'\, dm}^{\,0} + \frac{1}{2}\vec{\omega}\cdot\int \vec{r}'\times(\vec{\omega}\times\vec{r}')\, dm$$

$$= \frac{1}{2}\bar{v}^2 m + \vec{\bar{v}}\cdot\vec{\omega}\times\vec{0} + \frac{1}{2}\vec{\omega}\cdot\vec{H}_G = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\vec{\omega}\cdot\vec{H}_G$$

$$\vec{L} = m\vec{\bar{v}} \qquad \frac{1}{2}m\bar{v}^2 = \frac{1}{2}\vec{\bar{v}}\cdot m\vec{\bar{v}} = \frac{1}{2}\vec{\bar{v}}\cdot\vec{L}$$

Therefore, $\boxed{T = \frac{1}{2}\vec{\bar{v}}\cdot\vec{L} + \frac{1}{2}\vec{\omega}\cdot\vec{H}_G}$

In plane motion: $T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$

$$\boxed{T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\left(\bar{I}_{xx}\omega_x^2 + \bar{I}_{yy}\omega_y^2 + \bar{I}_{zz}\omega_z^2 - 2\bar{I}_{xy}\omega_x\omega_y - 2\bar{I}_{yz}\omega_y\omega_z - 2\bar{I}_{zx}\omega_z\omega_x\right)}$$

$\boxed{T = \frac{1}{2}I_{OA}\,\omega^2}$ where the body rotates about fixed axis OA.

Principle of work & kinetic energy: $\boxed{T_1 + U_{1\to 2} = T_2}$

(See Example 19.10 for illustration.)

Equations of motion for a rigid body

$$\boxed{\begin{aligned}\Sigma\vec{F} &= \dot{\vec{L}}\\ \Sigma\vec{M}_G &= \dot{\vec{H}}_G\\ \Sigma\vec{M}_O &= \dot{\vec{H}}_O\end{aligned}}$$

Euler's equations of motion:

$$\boxed{\begin{aligned}\Sigma M_x &= I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z\\ \Sigma M_y &= I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x\\ \Sigma M_z &= I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y\end{aligned}}$$