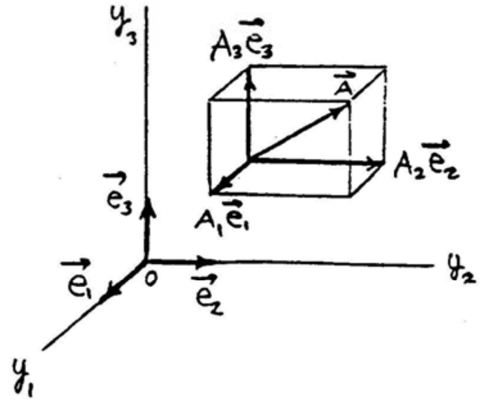
Index Notation & Cartesian Tensors

1. Cartesian Tensors

1-16-76

1.1 Index Notation

$$\vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k}$$
$$= A_1 \vec{e}_1 + A_2 \vec{e}_2 + A_3 \vec{e}_3$$
$$= \sum_{i=1}^{3} A_i \vec{e}_i \Rightarrow A_i \vec{e}_i \Rightarrow A_i$$



$$[B] = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \Rightarrow B_{ij}\ , \ \cdots$$

1.2 Range Convention.

Whenever a small Latin letter subscript occurs unrepeated in a term, it is understood to take on the values of 1, 2, 3.

$$A_i \Rightarrow (A_1\ \ A_2\ \ A_3)\ , \qquad \sigma_{ij} \Rightarrow \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}, \quad \cdots$$

1.3 Summation Convention.

Whenever a small Latin letter subscript occurs repeated in a term, it is understood to represent a summation over the range of 1, 2, 3.

$$A_i B_i = \sum_{i=1}^{3} A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3$$

$$\sigma_{ii} = \sum_{i=1}^{3} \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$\sigma_{ji} n_j = \sum_{j=1}^{3} \sigma_{ji} n_j = \sigma_{1i} n_1 + \sigma_{2i} n_2 + \sigma_{3i} n_3$$

$$\sigma_{ji,j} = \sum_{j=1}^{3} \sigma_{ji,j} = \sigma_{1i,1} + \sigma_{2i,2} + \sigma_{3i,3}$$

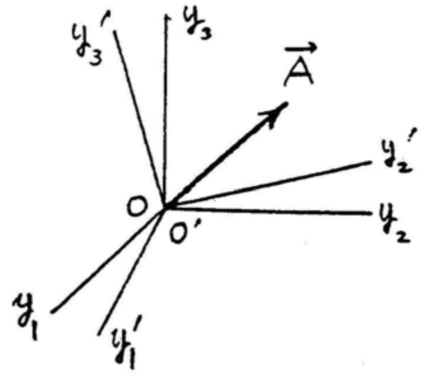
$$\vdots$$

1.4 Transformation Laws.

a_{ij} = cosine of the angle between the y'_i-axis and the y_j-axis.

j – Old or unprimed axis

i – New or primed axis



Scalar: $\phi'(y'_1, y'_2, y'_3) = \phi(y_1, y_2, y_3)$

Vector: $A'_i = a_{ij} A_j\,, \quad A_i = a_{ji} A'_j$

Bisor: $\sigma'_{ij} = a_{ik} a_{jl} \sigma_{kl}\,, \quad \sigma_{ij} = a_{ki} a_{lj} \sigma'_{kl}$

Trisor: $B'_{ijk} = a_{ir} a_{js} a_{kt} B_{rst}\,, \quad B_{ijk} = a_{ri} a_{sj} a_{tk} B'_{rst}$

Tetror: $C'_{ijkl} = a_{ir} a_{js} a_{kt} a_{lu} C_{rstu}$

$C_{ijkl} = a_{ri} a_{sj} a_{tk} a_{ul} C'_{rstu}$

Pentor, Hexor, Septor, Octor, ···

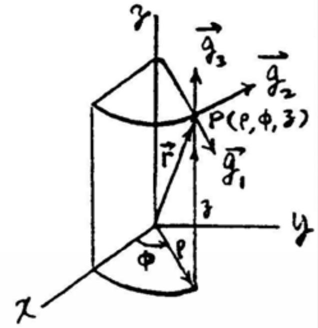
* 1.5 Base Vectors.

$\vec{r} = \vec{r}(x, y, z) = \vec{r}(x^1, x^2, x^3)$ ------ Position vector

$\vec{g}_i = \dfrac{\partial \vec{r}}{\partial x^i}$ -------- Covariant base vector

$\vec{g}^i = \vec{\nabla} x^i$ ------- Contravariant base vector

* 1.6 Illustration – Covariant and Contravariant Components of a Vector in Cylindrical Coordinates.



$\vec{r} = \rho \cos\phi\, \vec{i} + \rho \sin\phi\, \vec{j} + z \vec{k}$

$x^1 = \rho = (x^2 + y^2)^{\frac{1}{2}}, \quad x^2 = \phi = \tan^{-1} \dfrac{y}{x}, \quad x^3 = z = z$

$\vec{g}_1 = \cos\phi\, \vec{i} + \sin\phi\, \vec{j}, \quad \vec{g}_2 = -\rho \sin\phi\, \vec{i} + \rho \cos\phi\, \vec{j}, \quad \vec{g}_3 = \vec{k}$

$\vec{g}^1 = \cos\phi\, \vec{i} + \sin\phi\, \vec{j}, \quad \vec{g}^2 = \frac{1}{\rho}(-\sin\phi\, \vec{i} + \cos\phi\, \vec{j}), \quad \vec{g}^3 = \vec{k}$

$\vec{i} = \cos\phi\, \vec{g}_1 - \frac{1}{\rho} \sin\phi\, \vec{g}_2 = \cos\phi\, \vec{g}^1 - \rho \sin\phi\, \vec{g}^2$

$\vec{j} = \sin\phi\, \vec{g}_1 + \frac{1}{\rho} \cos\phi\, \vec{g}_2 = \sin\phi\, \vec{g}^1 + \rho \cos\phi\, \vec{g}^2$

$\vec{k} = \vec{g}_3 = \vec{g}^3 \qquad [x = \rho\cos\phi, \quad y = \rho \sin\phi, \quad z = z]$

3

For the vector $\vec{A} = z\vec{i} - 2x\vec{j} + y\vec{k}$ in cylindrical coordinates, it can be shown that

$$\vec{A} = A_1 \vec{g}^1 + A_2 \vec{g}^2 + A_3 \vec{g}^3 = A^1 \vec{g}_1 + A^2 \vec{g}_2 + A^3 \vec{g}_3$$

$$A_1 = z\cos\phi - 2\rho\sin\phi\cos\phi, \quad A_2 = -\rho z\sin\phi - 2\rho^2\cos^2\phi,$$

$$A_3 = \rho\sin\phi$$

$$A^1 = z\cos\phi - 2\rho\sin\phi\cos\phi, \quad A^2 = -\frac{1}{\rho} z\sin\phi - 2\cos^2\phi$$
$$A^3 = \rho\sin\phi$$

We see that in the present case

$$A_1 = A^1, \quad A_2 \neq A^2, \quad A_3 = A^3$$

$A_i \Rightarrow (A_1 \ A_2 \ A_3)$ ------ Covariant components of $\vec{A}$

$A^i \Rightarrow (A^1 \ A^2 \ A^3)$ ------ Contravariant components of $\vec{A}$

Exercise Knowing that $\vec{A} = y\vec{i} + z\vec{j} + x\vec{k}$, determine its covariant components $\bar{A}_1, \bar{A}_2, \bar{A}_3$, and its contravariant components $\bar{A}^1, \bar{A}^2, \bar{A}^3$ in cylindrical coordinates.

Show that $A_i = \frac{\partial \bar{x}^j}{\partial x^i} \bar{A}_j$:

$$A = \bar{A}_j \bar{g}^j = \bar{A}_j \nabla \bar{x}^j = \bar{A}_j \,\mathbf{e}_k \frac{\partial \bar{x}^j}{\partial y_k} = \bar{A}_j \,\mathbf{e}_k \frac{\partial \bar{x}^j}{\partial x^i}\frac{\partial x^i}{\partial y_k}$$

$$= \frac{\partial \bar{x}^j}{\partial x^i} \bar{A}_j \,\mathbf{e}_k \frac{\partial x^i}{\partial y_k} = \frac{\partial \bar{x}^j}{\partial x^i} \bar{A}_j \nabla x^i = \frac{\partial \bar{x}^j}{\partial x^i} \bar{A}_j g^i = A_i g^i$$

$$\left(A_i - \frac{\partial \bar{x}^j}{\partial x^i} \bar{A}_j\right) g^i = 0 \quad \text{for all } g^i\text{'s}$$

$$\therefore \; A_i - \frac{\partial \bar{x}^j}{\partial x^i} \bar{A}_j = 0 \qquad A_i = \frac{\partial \bar{x}^j}{\partial x^i} \bar{A}_j \qquad \text{Q.E.D.}$$

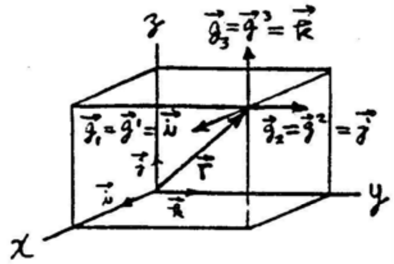
4

Supplementary Notes

1-23-76 10:30-11:20 a.m., E-110

* 1.7 Base Vectors in Cartesian Coordinates.

$$\begin{cases} \vec{r} = \vec{r}(x, y, z) = \vec{r}(x^1, x^2, x^3) \\ x^1 = x, \quad x^2 = y, \quad x^3 = z \\ \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \\ \quad = x^1\vec{i} + x^2\vec{j} + x^3\vec{k} \end{cases}$$



$$\begin{cases} \vec{g}_1 = \dfrac{\partial \vec{r}}{\partial x^1} = \vec{i}, \quad \vec{g}_2 = \dfrac{\partial \vec{r}}{\partial x^2} = \vec{j}, \quad \vec{g}_3 = \dfrac{\partial \vec{r}}{\partial x^3} = \vec{k} \\ \vec{g}^1 = \vec{\nabla} x^1 = \vec{\nabla} x = \vec{i}, \quad \vec{g}^2 = \vec{\nabla} x^2 = \vec{\nabla} y = \vec{j}, \quad \vec{g}^3 = \vec{\nabla} x^3 = \vec{\nabla} z = \vec{k} \end{cases}$$

$$\therefore \vec{g}_i = \vec{g}^i \Rightarrow (\vec{i} \;\; \vec{j} \;\; \vec{k})$$

* 1.8 Covariant and Contravariant Components of a Vector in Cartesian Coordinates.

$$\because \vec{A} = A_i \vec{g}^i = A^i \vec{g}_i \quad \text{and} \quad \vec{g}^i = \vec{g}_i$$

$\therefore A_i = A^i$ in Cartesian coordinates.

* 1.9 Transformation Laws of Covariant and Contravariant Tensors. $\vec{r} = \vec{r}(x^1, x^2, x^3) = \vec{r}(\bar{x}^1, \bar{x}^2, \bar{x}^3)$

Scalar: $\bar{\phi}(\bar{x}^1, \bar{x}^2, \bar{x}^3) = \phi(x^1, x^2, x^3)$

Vector: $\bar{A}_i = \dfrac{\partial x^j}{\partial \bar{x}^i} A_j$, $\quad A_i = \dfrac{\partial \bar{x}^j}{\partial x^i} \bar{A}_j$

$\bar{A}^i = \dfrac{\partial \bar{x}^i}{\partial x^j} A^j$, $\quad A^i = \dfrac{\partial x^i}{\partial \bar{x}^j} \bar{A}^j$

Bisor: $\bar{A}_{ij} = \dfrac{\partial x^k}{\partial \bar{x}^i} \dfrac{\partial x^l}{\partial \bar{x}^j} A_{kl}$, $\quad A_{ij} = \dfrac{\partial \bar{x}^k}{\partial x^i} \dfrac{\partial \bar{x}^l}{\partial x^j} \bar{A}_{kl}$

$\bar{A}^{ij} = \dfrac{\partial \bar{x}^i}{\partial x^k} \dfrac{\partial \bar{x}^j}{\partial x^l} A^{kl}$, $\quad A^{ij} = \dfrac{\partial x^i}{\partial \bar{x}^k} \dfrac{\partial x^j}{\partial \bar{x}^l} \bar{A}^{kl}$

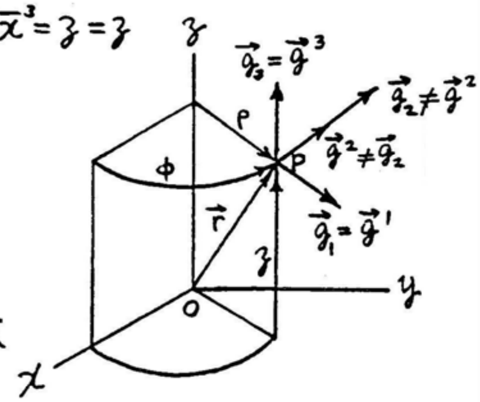
$\vdots$

* 1.10 Illustration – Covariant and Contravariant Components of a Vector in Cylindrical Coordinates (cf. Art. 1.6).

Let $\vec{A} = z\vec{i} - 2x\vec{j} + y\vec{k}$ be used again for illustration.

$$\begin{cases} \bar{x}^1 = \rho = (x^2+y^2)^{\frac{1}{2}}, \quad \bar{x}^2 = \phi = \tan^{-1}\dfrac{y}{x}, \quad \bar{x}^3 = z = z \\ x^1 = x, \quad x^2 = y, \quad x^3 = z \end{cases}$$

$$P(x, y, z) = P(\rho, \phi, z)$$



From Article 1.8, we know that the covariant and contravariant components of $\vec{A}$ are equal in Cartesian coordinates. Thus, we write

$$A_1 = A^1 = z, \quad A_2 = A^2 = -2x, \quad A_3 = A^3 = y$$

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z$$

The covariant and contravariant components of $\vec{A}$ in cylindrical coordinates are $\bar{A}_i$ and $\bar{A}^i$, respectively. By the transformation laws, we write

$$\begin{cases} \bar{A}_1 = \dfrac{\partial x^j}{\partial \bar{x}^1} A_j = \dfrac{\partial x^1}{\partial \bar{x}^1} A_1 + \dfrac{\partial x^2}{\partial \bar{x}^1} A_2 + \dfrac{\partial x^3}{\partial \bar{x}^1} A_3 = \dfrac{\partial x}{\partial \rho}(z) + \dfrac{\partial y}{\partial \rho}(-2x) + \dfrac{\partial z}{\partial \rho}(y) \\ \qquad = (\cos\phi)(z) + (\sin\phi)(-2\rho\cos\phi) + 0 = z\cos\phi - 2\rho\sin\phi\cos\phi \\ \bar{A}_2 = \dfrac{\partial x^j}{\partial \bar{x}^2} A_j = \dfrac{\partial x}{\partial \phi}(z) + \dfrac{\partial y}{\partial \phi}(-2x) + \dfrac{\partial z}{\partial \phi}(y) = -\rho z\sin\phi - 2\rho^2\cos^2\phi \\ \bar{A}_3 = \dfrac{\partial x^j}{\partial \bar{x}^3} A_j = \dfrac{\partial x}{\partial z}(z) + \dfrac{\partial y}{\partial z}(-2x) + \dfrac{\partial z}{\partial z}(y) = \rho\sin\phi \end{cases}$$

$$\begin{cases} \bar{A}^1 = \dfrac{\partial \bar{x}^1}{\partial x^j} A^j = \dfrac{\partial \bar{x}^1}{\partial x^1} A^1 + \dfrac{\partial \bar{x}^1}{\partial x^2} A^2 + \dfrac{\partial \bar{x}^1}{\partial x^3} A^3 = \dfrac{\partial \rho}{\partial x}(z) + \dfrac{\partial \rho}{\partial y}(-2x) + \dfrac{\partial \rho}{\partial z}(y) \\ \qquad = \dfrac{2x}{2\rho}(z) + \dfrac{2y}{2\rho}(-2x) + 0 = z\cos\phi - 2\rho\sin\phi\cos\phi \\ \bar{A}^2 = \dfrac{\partial \bar{x}^2}{\partial x^j} A^j = \dfrac{\partial \phi}{\partial x}(z) + \dfrac{\partial \phi}{\partial y}(-2x) + \dfrac{\partial \phi}{\partial z}(y) = \left(\dfrac{-y}{\rho^2}\right)(z) + \left(\dfrac{x}{\rho^2}\right)(-2x) + 0 \\ \qquad = -\dfrac{1}{\rho} z\sin\phi - 2\cos^2\phi \\ \bar{A}^3 = \dfrac{\partial \bar{x}^3}{\partial x^j} A^j = \dfrac{\partial z}{\partial z}(y) = \rho\sin\phi \end{cases}$$

* 1.11 Illustration — Physical Components of a Vector in Cylindrical Coordinates. (Not Tensors)

$$\vec{A} = \bar{A}_i \vec{g}^i = \bar{A}_1 \vec{g}^1 + \bar{A}_2 \vec{g}^2 + \bar{A}_3 \vec{g}^3 = \bar{A}_u \vec{e}_\rho + \bar{A}_v \vec{e}_\phi + \bar{A}_w \vec{e}_z$$

where $\vec{e}_\rho = \frac{\vec{g}^1}{|\vec{g}^1|}$, $\vec{e}_\phi = \frac{\vec{g}^2}{|\vec{g}^2|}$, $\vec{e}_z = \frac{\vec{g}^3}{|\vec{g}^3|}$

$(A_u\ A_v\ A_w)$ are the physical components in cylindrical coord.

From Art. 1.6, we write

$$\begin{cases} |\vec{g}^1|^2 = \vec{g}^1 \cdot \vec{g}^1 = (\cos\phi\, \vec{i} + \sin\phi\, \vec{j}) \cdot (\cos\phi\, \vec{i} + \sin\phi\, \vec{j}) = 1 \\ |\vec{g}^2|^2 = \vec{g}^2 \cdot \vec{g}^2 = \frac{1}{\rho}(-\sin\phi\, \vec{i} + \cos\phi\, \vec{j}) \cdot \left[\frac{1}{\rho}(-\sin\phi\, \vec{i} + \cos\phi\, \vec{j})\right] = \frac{1}{\rho^2} \\ |\vec{g}^3|^2 = \vec{g}^3 \cdot \vec{g}^3 = \vec{k} \cdot \vec{k} = 1 \end{cases}$$

$$\begin{cases} \bar{A}_u = \bar{A}_1 |\vec{g}^1| = \bar{A}_1, \quad \bar{A}_v = \bar{A}_2 |\vec{g}^2| = \frac{1}{\rho} A_2 \neq A_2 \\ \bar{A}_w = \bar{A}_3 |\vec{g}^3| = \bar{A}_3 \end{cases}$$

☆ $\bar{A}_i \Rightarrow (\bar{A}_1\ \bar{A}_2\ \bar{A}_3)$ <u>can</u> be related to $A_i \Rightarrow (A_1\ A_2\ A_3)$ according to the transformation law of tensors of the first order. $\bar{A}_i$ and A_i are covariant tensors of the first order.

☆ The set of physical components $(\bar{A}_u\ \bar{A}_v\ \bar{A}_w)$ <u>cannot</u> be related to the physical components $(A_u\ A_v\ A_w)$ according to the transformation law of tensors of the first order. Therefore, <u>physical components of vectors in cylindrical coordinates are not tensors</u>.

☆ In Cartesian coordinates, physical components are the same as tensor components.

2. Vector Algebra in Index Notations

1-30-76 Friday 10:30-11:20 a.m. E-110

2.1 Kronecker Delta δ_{ij}.

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \text{ numerically} \\ 0 & \text{if } i \neq j \text{ numerically} \end{cases} \qquad [\delta_{ij}] = \text{an identity matrix}$$

acting like a substituting symbol

$$\delta'_{ij} = \delta_{ij} = \delta_{ji} \qquad A_i = \delta_{ij} A_j$$

(Substituting the unrepeated for the repeated)

2.2 General Rules.

Rule 1 : In general, do not repeat an index in a term more than one time.

Rule 2 : In a tensor equation, the range indices must be the same in each term. (The homogeneity of range indices.)

2.3 Orthonormal Conditions.

$$A_i = a_{ki} A'_k = a_{ki}(a_{kj} A_j) = a_{ki} a_{kj} A_j = \delta_{ij} A_j$$

$$(a_{ki} a_{kj} - \delta_{ij}) A_j = 0, \qquad \underline{a_{ki} a_{kj} = \delta_{ij}}$$

$$A'_i = a_{ik} A_k = a_{ik}(a_{jk} A'_j) = a_{ik} a_{jk} A'_j = \delta_{ij} A'_j$$

$$(a_{ik} a_{jk} - \delta_{ij}) A'_j = 0, \qquad \underline{a_{ik} a_{jk} = \delta_{ij}}$$

(The definition of a_{ij} was given in Art. 1.4.)

2.4 Tensorial Character of δ_{ij}.

$$\because \; a_{ik} a_{jl} \delta_{kl} = a_{ik} a_{jk} = \delta_{ij} = \delta'_{ij}$$

$$\therefore \; \delta'_{ij} = a_{ik} a_{jl} \delta_{kl} \quad \text{-------- Second order tensor}$$

8

2.5 Addition and Subtraction.

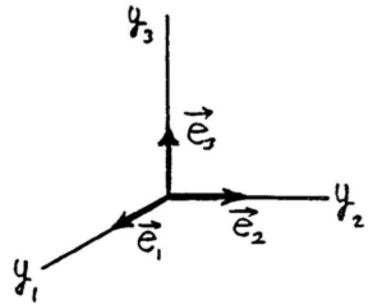
The addition or subtraction of Cartesian tensors of the same order is to be carried out for their corresponding components.

$$A_{ij\cdots} \pm B_{ij\cdots} = C_{ij\cdots}$$

where Rule 2 in Art. 2.2 is observed.

2.6 Scalar (or Dot) Product of Basic Unit Vectors $\vec{e}_i$.

$$\vec{e}_1 \cdot \vec{e}_2 = \vec{e}_2 \cdot \vec{e}_1 = 0$$
$$\vec{e}_2 \cdot \vec{e}_3 = \vec{e}_3 \cdot \vec{e}_2 = 0$$
$$\vec{e}_3 \cdot \vec{e}_1 = \vec{e}_1 \cdot \vec{e}_3 = 0$$
$$\vec{e}_1 \cdot \vec{e}_1 = \vec{e}_2 \cdot \vec{e}_2 = \vec{e}_3 \cdot \vec{e}_3 = 1$$
$$\therefore \; \vec{e}_i \cdot \vec{e}_j = \delta_{ij}$$



2.7 Scalar Product of Two Vectors.

$$\vec{A} \cdot \vec{B} = (A_i \vec{e}_i) \cdot (B_j \vec{e}_j) = A_i B_j (\vec{e}_i \cdot \vec{e}_j) = A_i B_j \delta_{ij}$$
$$\therefore \; \vec{A} \cdot \vec{B} = A_i B_i$$

2.8 Permutation Symbol ϵ_{ijk}.

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ form an even permutation of } 123, \\ -1 & \text{" " " " odd " " ",} \\ 0 & \text{otherwise.} \end{cases}$$

$$\epsilon_{123} = 1, \quad \epsilon_{231} = -\epsilon_{213} = \epsilon_{123} = 1, \quad \epsilon_{213} = -\epsilon_{123} = -1,$$
$$\epsilon_{312} = -\epsilon_{132} = \epsilon_{123} = 1, \quad \epsilon_{112} = 0, \quad \epsilon_{222} = 0, \; \cdots$$
$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki}$$

In Cartesian coordinates, it can be shown that

$$\epsilon'_{ijk} = a_{ir} a_{js} a_{kt} \epsilon_{rst} \quad \text{------ Third order tensor}$$

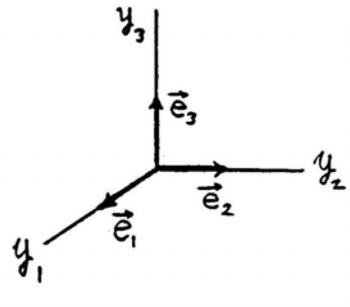
2.9 Vector (or Cross) Product of Basic Unit Vectors $\vec{e}_i$

$$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3 \,, \quad \vec{e}_2 \times \vec{e}_1 = -\vec{e}_3$$

$$\vec{e}_2 \times \vec{e}_3 = \vec{e}_1 \,, \quad \vec{e}_3 \times \vec{e}_2 = -\vec{e}_1$$

$$\vec{e}_3 \times \vec{e}_1 = \vec{e}_2 \,, \quad \vec{e}_1 \times \vec{e}_3 = -\vec{e}_2$$

$$\therefore \ \vec{e}_i \times \vec{e}_j = \epsilon_{ijk} \vec{e}_k$$



2.10 Vector Product of Two Vectors.

$$\vec{A} \times \vec{B} = (A_j \vec{e}_j) \times (B_k \vec{e}_k) = A_j B_k (\vec{e}_j \times \vec{e}_k)$$

$$= A_j B_k \epsilon_{jki} \vec{e}_i = A_j B_k \epsilon_{ijk} \vec{e}_i = \epsilon_{ijk} A_j B_k \vec{e}_i$$

$$\therefore \ \vec{A} \times \vec{B} \Rightarrow \epsilon_{ijk} A_j B_k$$

2.11 Scalar Triple Product.

From Art. 2.10, we have

$$\vec{A} \times \vec{B} \cdot \vec{C} = (\epsilon_{ijk} A_j B_k \vec{e}_i) \cdot (C_l \vec{e}_l)$$

$$= \epsilon_{ijk} A_j B_k C_l (\vec{e}_i \cdot \vec{e}_l) = \epsilon_{ijk} A_j B_k C_l \delta_{il}$$

$$= \epsilon_{ijk} A_j B_k C_i = \epsilon_{jki} A_j B_k C_i = \epsilon_{ijk} A_i B_j C_k$$

$$\therefore \ \vec{A} \cdot \vec{B} \times \vec{C} = \vec{A} \times \vec{B} \cdot \vec{C} = [\vec{A}\,\vec{B}\,\vec{C}] = \epsilon_{ijk} A_i B_j C_k$$

$$\epsilon_{lmn} = \delta_{lr} \delta_{ms} \delta_{nt} \epsilon_{rst} = (a_{il} a_{jm} a_{kn})(a_{ir} a_{js} a_{kt} \epsilon_{rst})$$

$$= (a_{il} a_{jm} a_{kn})(\epsilon'_{ijk}) = a_{il} a_{jm} a_{kn} \epsilon'_{ijk}$$

If $\epsilon'_{ijk} = a_{ir} a_{js} a_{kt} \epsilon_{rst}$, then $\epsilon_{lmn} = a_{il} a_{jm} a_{kn} \epsilon'_{ijk}$.

10

2-6-76 10:30–11:20 a.m.
E-110

3. Gradient, Divergence, and Curl

3.1 Gradient of a Scalar Function.

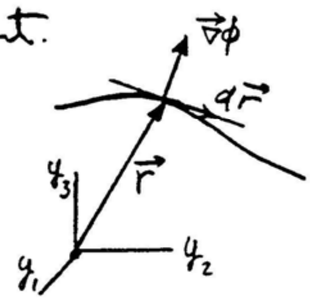
$$\vec{\nabla} = \overrightarrow{grad} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} = \vec{e}_1\frac{\partial}{\partial y_1} + \vec{e}_2\frac{\partial}{\partial y_2} + \vec{e}_3\frac{\partial}{\partial y_3}$$

$$= \vec{e}_i\frac{\partial}{\partial y_i} = \vec{e}_i\,\partial_i \Rightarrow \partial_i \quad \text{or} \quad \boxed{\vec{\nabla} \Rightarrow \frac{\partial}{\partial y_i} = \partial_i}$$

$$\vec{\nabla}\phi(x, y, z) = \vec{\nabla}\phi(y_1, y_2, y_3) = \vec{\nabla}\phi = \vec{e}_i\frac{\partial\phi}{\partial y_i} \Rightarrow \frac{\partial\phi}{\partial y_i}$$

$$\underline{\vec{\nabla}\phi \Rightarrow \frac{\partial\phi}{\partial y_i} = \partial_i\phi = \phi_{,i} \Rightarrow (\phi_{,1} \quad \phi_{,2} \quad \phi_{,3}) \Rightarrow \overrightarrow{grad}\,\phi}$$

Note that $\vec{\nabla}\phi$ is a vector perpendicular to the surface given by $\phi = \phi(y_1, y_2, y_3) = $ constant.



$$\vec{\nabla}\phi \cdot d\vec{r} = \left(\vec{e}_i\frac{\partial\phi}{\partial y_i}\right)\cdot\left(\vec{e}_j\,dy_j\right)$$

$$= \vec{e}_i\cdot\vec{e}_j\,\frac{\partial\phi}{\partial y_i}dy_j = \delta_{ij}\frac{\partial\phi}{\partial y_i}dy_j$$

$$= \frac{\partial\phi}{\partial y_i}dy_i = d\phi = d(\text{const.}) = 0$$

Q. E. D.

3.2 Divergence of a Vector Function.

$$\vec{\nabla}\cdot\vec{A} = \left(\vec{e}_i\frac{\partial}{\partial y_i}\right)\cdot\left(A_j\vec{e}_j\right) = \left(\vec{e}_i\cdot\vec{e}_j\right)\frac{\partial A_j}{\partial y_i} = \delta_{ij}A_{j,i}$$

$$\underline{\vec{\nabla}\cdot\vec{A} = A_{i,i} = \frac{\partial A_1}{\partial y_1} + \frac{\partial A_2}{\partial y_2} + \frac{\partial A_3}{\partial y_3} = \text{div}\,\vec{A}}$$

Physically, $\vec{\nabla}\cdot\vec{A}$ at the point P represents the rate of decrease of the flux of $\vec{A}$ per unit volume at the point P.

11

3.3 Laplacian Operator ∇^2.

$$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \left(\vec{e}_i \frac{\partial}{\partial y_i}\right)\cdot\left(\vec{e}_j \frac{\partial}{\partial y_j}\right) = (\vec{e}_i\cdot\vec{e}_j)\frac{\partial^2}{\partial y_i \partial y_j}$$

$$= \delta_{ij}\frac{\partial^2}{\partial y_i \partial y_j} = \frac{\partial^2}{\partial y_i \partial y_i} = \partial_i \partial_i = (\quad)_{,ii}$$

$$\underline{\nabla^2 = \partial_i \partial_i = \frac{\partial^2}{\partial y_i \partial y_i} = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2}}$$

∇^2 is a scalar differential operator.

3.4 Curl of a Vector Function.

$$\vec{\nabla}\times\vec{A} = \left(\vec{e}_j \frac{\partial}{\partial y_j}\right)\times(A_k \vec{e}_k) = (\vec{e}_j \times \vec{e}_k)\frac{\partial A_k}{\partial y_j}$$

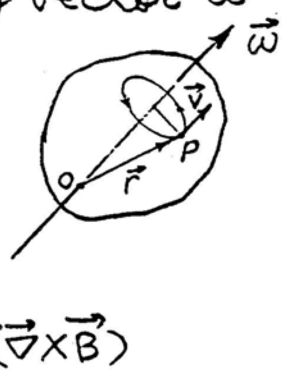
$$= (\epsilon_{jki}\vec{e}_i)A_{k,j} = \vec{e}_i\, \epsilon_{ijk} A_{k,j}$$

$$= \vec{e}_i\, \epsilon_{ijk}\partial_j A_k \Rightarrow \epsilon_{ijk}\partial_j A_k$$

$$\underline{\vec{\nabla}\times\vec{A} \Rightarrow \epsilon_{ijk}\partial_j A_k = \epsilon_{ijk}A_{k,j} = \text{curl } \vec{A}}$$

Physically, the curl of the linear velocity vector $\vec{V}$ of a point on a rigid body is equal to twice of the angular velocity vector $\vec{\omega}$ of the rigid body; i.e.,

$$\vec{\nabla}\times\vec{V} = \vec{\nabla}\times(\vec{\omega}\times\vec{r}) = 2\vec{\omega}$$



3.5 Illustration – Prove the identity

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

Proof: The identity to be proved may be written as

$$\partial_i (\epsilon_{ijk} A_j B_k) = B_i \epsilon_{ijk} \partial_j A_k - A_i \epsilon_{ijk} \partial_j B_k$$

$$\begin{aligned}
\text{LHS} &= \epsilon_{ijk} (A_{j,i} B_k + A_j B_{k,i}) \\
&= \epsilon_{ijk} A_{j,i} B_k + \epsilon_{ijk} A_j B_{k,i} \\
\text{RHS} &= \epsilon_{ijk} A_{k,j} B_i - \epsilon_{ijk} A_i B_{k,j} \\
&= \epsilon_{kij} A_{j,i} B_k - \epsilon_{jik} A_j B_{k,i} \\
&= \epsilon_{ijk} A_{j,i} B_k + \epsilon_{ijk} A_j B_{k,i}
\end{aligned}$$

Q. E. D.

3.6 __Illustration__ — Prove the identity $\vec{\nabla} \times (\vec{\nabla}\phi) = \vec{0}$.

__Proof__: The identity to be proved may be written as

$$\epsilon_{ijk} \partial_j (\partial_k \phi) = 0$$

$$\begin{aligned}
\text{LHS} &= \epsilon_{ijk} \partial_j (\phi_{,k}) = \epsilon_{ijk} \phi_{,kj} = \epsilon_{ijk} \phi_{,jk} \\
&= -\epsilon_{ikj} \phi_{,jk} = -\epsilon_{ijk} \phi_{,kj}
\end{aligned}$$

$\therefore \; 2\epsilon_{ijk} \phi_{,kj} = 0, \qquad \text{LHS} = 0. \qquad$ Q. E. D.

3.7 __Exercise__ — Prove the identity $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$.

2-13-76 10:30–11:20 a.m.
E-110

4. ϵ-δ Indentity and Its Applications

4.1 Generalized Kronecker Delta δ^{ijk}_{rst}.

$$\delta^{ijk}_{rst} = \epsilon_{ijk}\,\epsilon_{rst} = \begin{vmatrix} \delta_{ir} & \delta_{jr} & \delta_{kr} \\ \delta_{is} & \delta_{js} & \delta_{ks} \\ \delta_{it} & \delta_{jt} & \delta_{kt} \end{vmatrix} = \begin{vmatrix} \delta_{ir} & \delta_{is} & \delta_{it} \\ \delta_{jr} & \delta_{js} & \delta_{jt} \\ \delta_{kr} & \delta_{ks} & \delta_{kt} \end{vmatrix}$$

$$= \delta_{ir}\delta_{js}\delta_{kt} + \delta_{jr}\delta_{ks}\delta_{it} + \delta_{kr}\delta_{is}\delta_{jt} - \delta_{kr}\delta_{js}\delta_{it}$$
$$- \delta_{jr}\delta_{is}\delta_{kt} - \delta_{ir}\delta_{ks}\delta_{jt}$$

4.2 The "ϵ-δ" Identity.

$$\epsilon_{ijk}\,\epsilon_{irs} = \epsilon_{jki}\,\epsilon_{irs} = \delta^{ijk}_{irs} = \begin{vmatrix} \delta_{ii} & \delta_{ji} & \delta_{ki} \\ \delta_{ir} & \delta_{jr} & \delta_{kr} \\ \delta_{is} & \delta_{js} & \delta_{ks} \end{vmatrix}$$

$$= \delta_{ii}\delta_{jr}\delta_{ks} + \delta_{ji}\delta_{kr}\delta_{is} + \delta_{ki}\delta_{ir}\delta_{js} - \delta_{ki}\delta_{jr}\delta_{is}$$
$$- \delta_{ji}\delta_{ir}\delta_{ks} - \delta_{ii}\delta_{kr}\delta_{js}$$

$$= 3\,\delta_{jr}\delta_{ks} + \delta_{kr}\delta_{js} + \delta_{kr}\delta_{js} - \delta_{jr}\delta_{ks} - \delta_{jr}\delta_{ks} - 3\delta_{kr}\delta_{js}.$$

$$= \delta_{jr}\delta_{ks} - \delta_{kr}\delta_{js} = \delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr}$$

$$\boxed{\epsilon_{ijk}\,\epsilon_{irs} = \epsilon_{jki}\,\epsilon_{irs} = \delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr} = \delta^{ijk}_{irs}}$$

4.3 Illustration — Prove the identity

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}\,(\vec{C}\cdot\vec{A}) - \vec{C}\,(\vec{A}\cdot\vec{B})$$

<u>Proof</u>: The identity to be proved may be written as

$$\epsilon_{ijk} A_j \left(\epsilon_{klm} B_l C_m \right) = B_i \left(C_j A_j \right) - C_i \left(A_j B_j \right)$$

$$\text{LHS} = \epsilon_{ijk} \epsilon_{klm} A_j B_l C_m = \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right) A_j B_l C_m$$

$$= \delta_{il} \delta_{jm} A_j B_l C_m - \delta_{im} \delta_{jl} A_j B_l C_m$$

$$= A_m B_i C_m - A_l B_l C_i$$

$$\text{RHS} = A_j B_i C_j - A_j B_j C_i = A_m B_i C_m - A_l B_l C_i$$

Q. E. D.

4.4 <u>Illustration</u> — Prove the identity

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} \right) = \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} \right) - \nabla^2 \vec{A}$$

<u>Proof</u>: The identity to be proved may be written as

$$\epsilon_{ijk} \partial_j \left(\epsilon_{klm} \partial_l A_m \right) = \partial_i \left(\partial_j A_j \right) - \partial_j \partial_j A_i$$

$$\text{LHS} = \epsilon_{ijk} \epsilon_{klm} \partial_j \partial_l A_m = \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right) A_{m,lj}$$

$$= A_{j,ij} - A_{i,jj}$$

$$\text{RHS} = A_{j,ji} - A_{i,jj} = A_{j,ij} - A_{i,jj}$$

Q. E. D.

4.5 <u>Illustration</u> — Prove the identity

$$\vec{\nabla} \times \left(\vec{A} \times \vec{B} \right) = \left(\vec{B} \cdot \vec{\nabla} \right) \vec{A} - \left(\vec{A} \cdot \vec{\nabla} \right) \vec{B} + \left(\vec{\nabla} \cdot \vec{B} \right) \vec{A} - \left(\vec{\nabla} \cdot \vec{A} \right) \vec{B}$$

<u>Proof</u>: The identity to be proved may be written as

$$\epsilon_{ijk}\partial_j(\epsilon_{klm}A_l B_m) = (B_j\partial_j)A_i - (A_j\partial_j)B_i + (\partial_j B_j)A_i - (\partial_j A_j)B_i$$

$$\text{LHS} = \epsilon_{ijk}\epsilon_{klm}\partial_j(A_l B_m)$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(A_{l,j}B_m + A_l B_{m,j})$$

$$= A_{i,m}B_m - A_{j,j}B_i + A_i B_{j,j} - A_j B_{i,j}$$

$$\text{RHS} = B_j A_{i,j} - A_j B_{i,j} + B_{j,j}A_i - A_{j,j}B_i$$

$$= A_{i,m}B_m - A_j B_{i,j} + A_i B_{j,j} - A_{j,j}B_i$$

$$= A_{i,m}B_m - A_{j,j}B_i + A_i B_{j,j} - A_j B_{i,j}$$

Q. E. D.

16