Kindling Students' Interest in Virtual Work Method: Advantages and Challenges

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Abstract

In mechanics, work is energy in transition to a body due to force or moment acting on the body through a displacement. Like heat, work is a scalar quantity occurring during a process. A virtual work is the work done by a force or moment due to a virtual displacement of the body. By giving a compatible virtual displacement for the system and applying the principle of virtual work, we can solve many complex as well as simple problems in mechanics without a need to solve coupled simultaneous equations. This powerful feature of the virtual work method may initially appear as a magic black box to students, but it kindles great interest and excitement in them. This paper reviews the fundamentals of the virtual work method, compares it with the conventional method, and points out the inherent advantages and challenges. Seemingly challenging examples in addition to simple examples are included.

I. Fundamental Concepts

Engineering and technology students learn the definition of *work* when they take the course in physics usually in their freshman year. In mechanics, a body receives work from a force or a moment that acts on it if it undergoes a displacement in the direction of the force or moment, respectively, during the action. It is the force or moment, rather than the body, which does work. Before reaping advantages in the virtual work method, it is well to refresh certain fundamental concepts, which sometimes appear as challenges to beginning students.

Work of a force.

The **work** $U_{1\to 2}$ done by a force $\mathbf{F}$ on a body moving from position A_1 along a path C to position A_2 is defined by a line integral. It is given by [1-4]

$$U_{1\to 2} = \int_{A_1}^{A_2} \mathbf{F}\cdot d\mathbf{r} \tag{1}$$

where $\cdot$ denotes a dot product, and $d\mathbf{r}$ is the differential displacement of the body moving along the path C during the action of $\mathbf{F}$ on the body. If the force $\mathbf{F}$ is *constant* and the displacement vector of the body during the action is $\mathbf{q}$, then the work done on the body is given by

$$U_{1\to 2} = \mathbf{F}\cdot\mathbf{q} = Fq_{\parallel} \tag{2}$$

where F is the magnitude of $\mathbf{F}$ and $q_{\parallel}$ is the scalar component of $\mathbf{q}$ parallel to the force $\mathbf{F}$. If we let the angle between the positive directions of $\mathbf{F}$ and $\mathbf{q}$ be ϕ and assume that both $\mathbf{F}$ and $\mathbf{q}$ are not zero, then notice that the values of both $q_{\parallel}$ and $U_{1\to 2}$ are negative if and only if $90° < \phi \le 180°$.

Work of a moment.

The **work** $U_{1\to 2}$ done by a moment **M** (or a couple of moment **M**) on a body during its finite rotation, parallel to **M**, from angular position θ_1 to angular position θ_2 is given by [1-4]

$$U_{1\to 2} = \int_{\theta_1}^{\theta_2} M\, d\theta \tag{3}$$

If the moment **M** is *constant* and the angular displacement of the body in the direction of **M** during the action is $\Delta\theta$, then the work done on the body is given by

$$U_{1\to 2} = M(\Delta\theta) \tag{4}$$

Compatible virtual displacement.

An *actual displacement* of a body, as the name implies, indicates the actual change of the position of the body. A **virtual displacement** of a body is a given or imaginary differential displacement, which is possible but does not necessarily take place under actual motion. There are *linear* and *angular virtual displacements*; they are vector quantities. A **compatible virtual displacement** of a body is a set of imaginary first-order differential displacements, which conforms to the integrity (i.e., no breakage or rupture) of its free body within the framework of first-order accuracy, where the body may be a particle, a rigid body, or a set of connected particles or rigid bodies. Note that a *compatible virtual displacement* of a body does not necessarily conform to the constraints at the supports in the space diagram of the body.

Displacement center and radian measure.

Relations among the virtual displacements of certain points or members in a system can be found by using *differential calculus*, or the *displacement center*,[5] or both. The **displacement center** of a body (or a member in a system) is the center of rotation of the body when it undergoes an angular virtual displacement and the points on it incur linear virtual displacements. Such a center is located at (*a*) the point of zero displacement on the body, or (*b*) the intersection of two straight lines that pass through two different points of the body and are perpendicular to the linear virtual displacement vectors of those two points, respectively. It is important to know the location of the displacement center for each member corresponding to a chosen set of compatible virtual displacements for the free body of the system. Once the displacement centers are known, the linear virtual displacements of the points and the angular virtual displacements of the members can readily be found in terms of a single variable using mainly simple geometric and algebraic relations, as well as the **radian measure** formula ($\delta s = r\,\delta\theta$, where δs is the arc subtending an angle $\delta\theta$ in radian included by two radii of length r), rather than calculus.

Principle of virtual work.

Historical studies show that on February 26, 1715, the Swiss mathematician Johann Bernoulli (1667-1748) communicated to Pierre Varignon (1654-1722) the principle of virtual velocities in analytical form for the first time. That was the forerunner of the principle of virtual work today. The approach to mechanics based on the principle of virtual work was formally treated by Joseph Louis Lagrange (1736-1813) in his *Mécanique Analytique* published in 1788. For a system in equilibrium, the resultant force and the resultant moment acting on the free body of the system are both equal to zero. This fact provides a basis that leads to the **principle of virtual work** in statics, which may be stated as follows: *If a body is in equilibrium, the total virtual work of the external force system acting on its free body during any compatible virtual displacement of its free body is equal to zero, and conversely*. Note that the body in this principle may be a particle, a set of connected particles, a rigid body, a frame, or a machine.

II. Conventional Method versus Virtual Work Method in Solving a Simple Problem

The **conventional method** for solving equilibrium problems uses two basic equilibrium equations: (*a*) force equilibrium equation, and (*b*) moment equilibrium equation; i.e.,

$$\Sigma \mathbf{F} = \mathbf{0} \tag{5}$$

$$\Sigma \mathbf{M}_P = \mathbf{0} \tag{6}$$

The **virtual work method** is used to solve equilibrium problems by setting to zero the total virtual work δU done by the force system on the free body corresponding to a chosen set of compatible virtual displacements of the free body; i.e.,

$$\delta U = 0 \tag{7}$$

Example 1. First use the *conventional method* then use the *virtual work method* to determine the reactions at the supports *A* and *B* of the beam loaded as shown in Fig. 1.

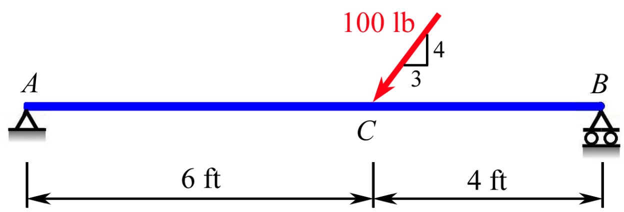

Fig. 1 A simple beam carrying an inclined concentrated load

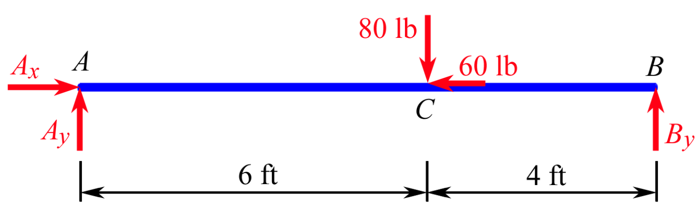

Fig. 2 Free-body diagram for the beam in Fig. 1

- **Solution by the conventional method:** Referring to the free-body diagram in Fig. 2 and applying Eq. (6), as well as Eq. (5), we write

$$+\circlearrowleft \Sigma M_A = 0: \quad 10B_y - 6(80) = 0 \quad \therefore \quad B_y = 48$$

$$\overset{+}{\rightarrow} \Sigma F_x = 0: \quad A_x - 60 = 0 \quad \therefore \quad A_x = 60$$

$$+\uparrow \Sigma F_y = 0: \quad A_y - 80 + B_y = 0 \quad \therefore \quad A_y = 32$$

Thus, the reactions at *A* and *B*, respectively, are

$$\mathbf{A} = 60\mathbf{i} + 32\mathbf{j} \text{ lb} \qquad \mathbf{B} = 48\mathbf{j} \text{ lb}$$

Solution by the virtual work method: To solve for any unknown appearing in the free-body diagram in Fig. 2, we first draw a compatible virtual displacement for the beam with a *strategy* in such a way that no unknowns *except the one to be solved* will be involved in the total virtual work done.

- To solve for the unknown B_y, we draw a *virtual-displacement diagram* as shown in Fig. 3.

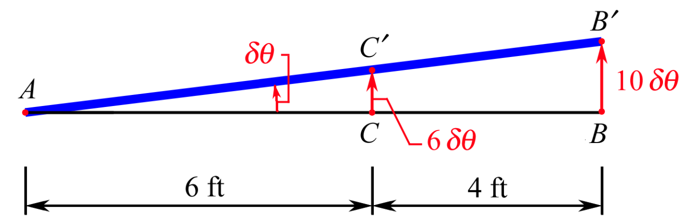

Fig. 3 Virtual-displacement diagram for use in solving for B_y

Referring to Figs. 2 and 3 and applying the virtual work principle, we write

$$\delta U = 0: \qquad B_y\,(10\,\delta\theta) + 80\,(-\,6\;\delta\theta) = 0 \qquad \therefore \quad B_y = 48$$

- To solve for the unknown A_x, we draw a *virtual-displacement diagram* as shown in Fig. 4.

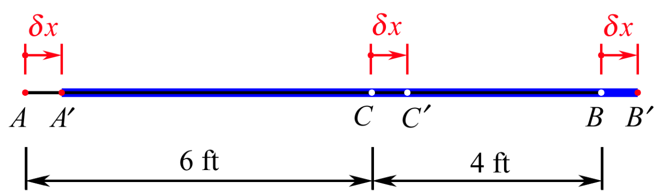

Fig. 4 Virtual-displacement diagram for use in solving for A_x

Referring to Figs. 2 and 4 and applying the virtual work principle, we write

$$\delta U = 0: \qquad A_x\,(\delta x\,) + 60\,(-\,\delta x\,) = 0 \qquad \therefore \quad A_x = 60$$

- To solve for the unknown A_y, we draw a *virtual-displacement diagram* as shown in Fig. 5.

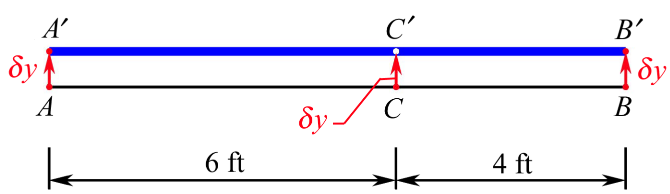

Fig. 5 Virtual-displacement diagram for use in solving for A_y

Referring to Figs. 2 and 5 and applying the virtual work principle, we write

$$\delta U = 0: \qquad A_y\,(\delta y) + 80\,(-\,\delta y) + B_y\,(\delta y) = 0 \qquad \therefore \quad A_y = 32$$

Thus, the reactions at A and B, respectively, are

$$\mathbf{A} = 60\,\mathbf{i} + 32\,\mathbf{j}\ \text{lb} \qquad\qquad \mathbf{B} = 48\,\mathbf{j}\ \text{lb}$$

which are identical with the reactions obtained earlier by the conventional method.

III. Advantages in Virtual Work Method: Examples

In solving a simple problem, the solution by the virtual work method seems to come across as more “cumbersome” or even “funny” when it is compared with the solution by the conventional method as shown in Example 1 in Section II. Well, it should be noted that Example 1 was intended to serve merely as a learning example for beginning students to show how the steps in the solution by the virtual work method compare with those in the solution by the conventional method. The virtual work method is a powerful method that will show big **advantages** over the conventional method when more complex problems are to be solved. In other words, the virtual work method is an effective and serious (not “cumbersome” or “funny”) method in solving decently complex problems as illustrated below.

Example 2. Determine reaction moment $\mathbf{M}_A$ at the fixed support A of the *Gerber beam* loaded as shown in Fig. 6.

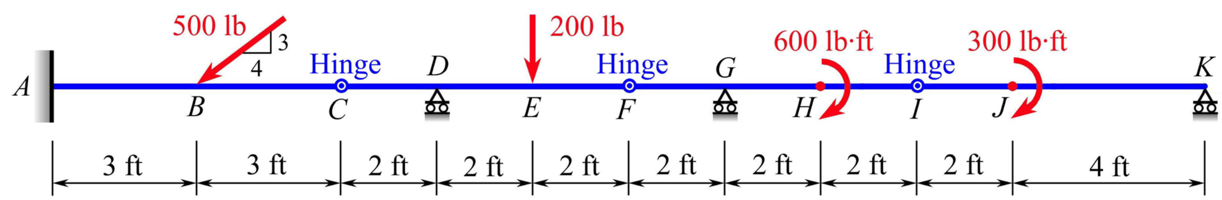

Fig. 6 A *Gerber beam* with hinge connections at C, F, and I

Solution. We first draw the free-body diagram and a set of compatible virtual displacements for the beam as shown in Fig. 7. Note that we draw this virtual-displacement diagram with a *strategy* such that no unknowns *except* M_A will be involved in the total virtual work done.

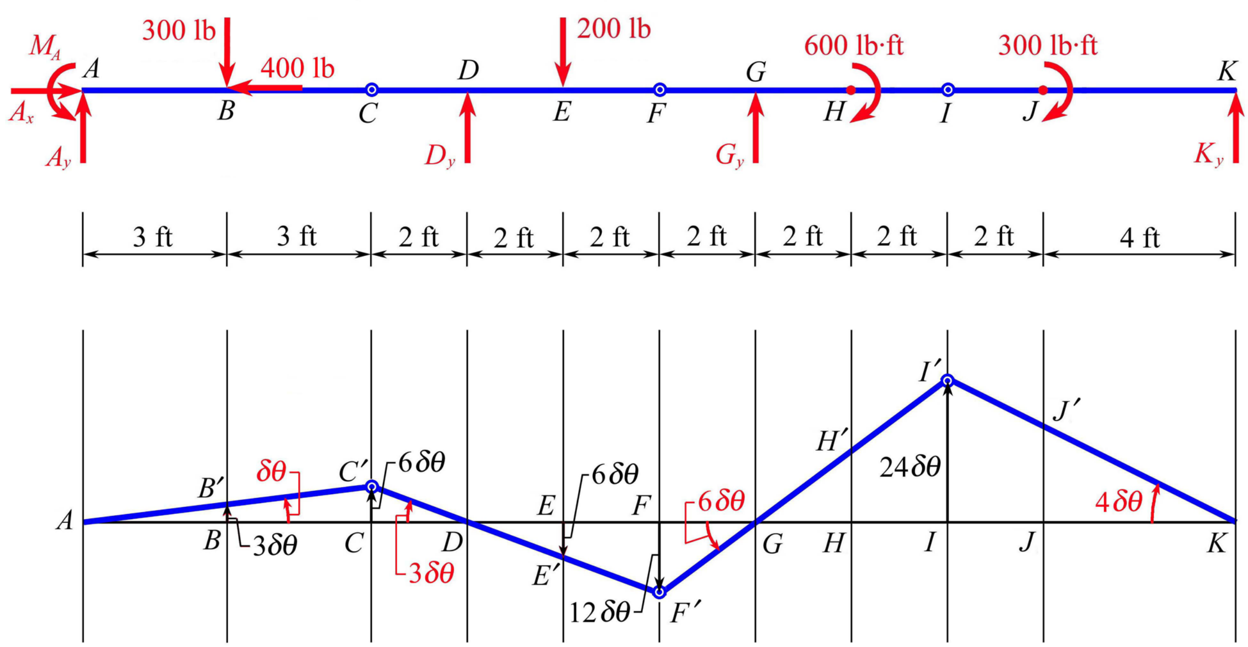

Fig. 7 Free-body diagram and virtual-displacement diagram for use in solving for M_A

Referring to Fig. 7 and applying the virtual work principle, we write

$$\delta U = 0: \qquad M_A(\delta\theta) + 300(-3\delta\theta) + 200(6\delta\theta) + 600(-6\delta\theta) + 300(4\delta\theta) = 0$$

$$M_A = 2100 \qquad\qquad \mathbf{M}_A = 2100 \text{ lb·ft } \circlearrowleft$$

Remarks. Notice that the free-body diagram for this *Gerber beam* contains **six** unknowns. The above solution by the *virtual work method* shows that we are able to solve for the designated unknown M_A without the need to solve simultaneous equations! This advantage may initially appear as a magic black box to students and cannot be matched by the solution using the conventional method.

Example 3. Determine the vertical reaction force $\mathbf{A}_y$ at the fixed support A of the *Gerber beam* shown in Fig. 6.

Solution. We first draw the free-body diagram and a set of compatible virtual displacements for the beam as shown in Fig. 8. Note that we draw this virtual-displacement diagram with a *strategy* such that no unknowns *except* A_y will be involved in the total virtual work done.

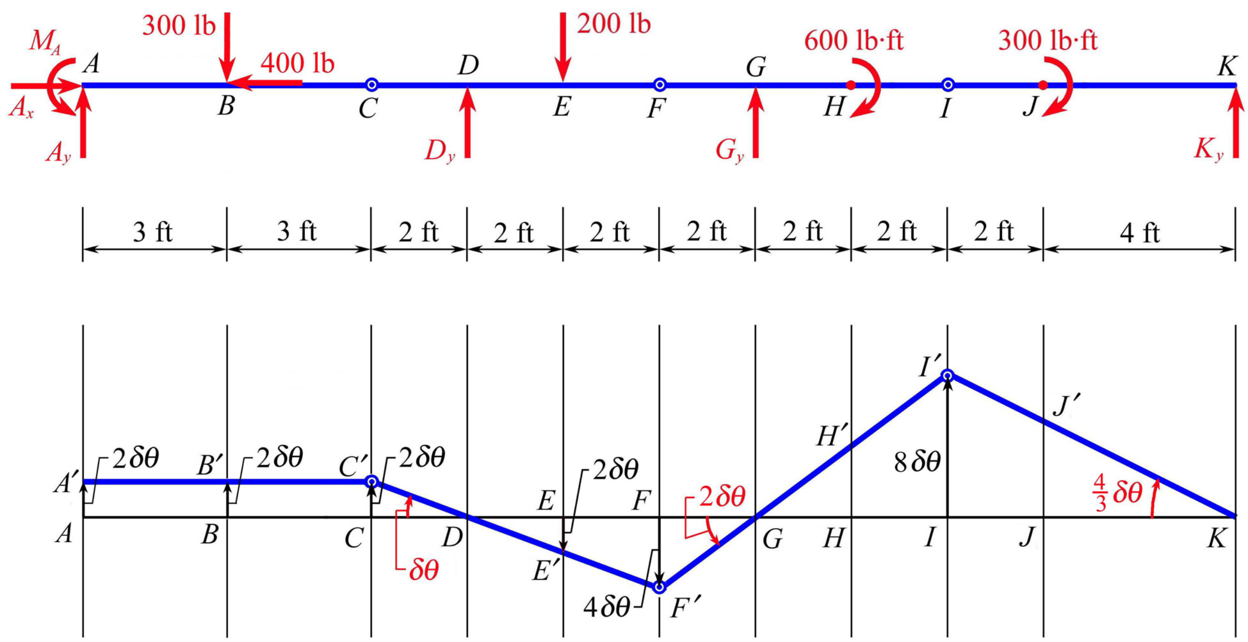

Fig. 8 Free-body diagram and virtual-displacement diagram for use in solving for A_y

Referring to Fig. 8 and applying the virtual work principle, we write

$$\delta U = 0: \qquad A_y(2\delta\theta) + 300(-2\delta\theta) + 200(2\delta\theta) + 600(-2\delta\theta) + 300\left(\tfrac{4}{3}\delta\theta\right) = 0$$

$$A_y = 500 \qquad\qquad \mathbf{A}_y = 500 \text{ lb} \uparrow$$

Remarks. There it goes again. The above solution by the *virtual work method* shows that we are able to solve for the designated unknown A_y without the need to solve simultaneous equations!

This advantage may initially appear as a magic black box to students and cannot be matched by the solution using the conventional method.

Example 4. Determine the horizontal reaction force $\mathbf{D}_x$ at the fixed support D of the frame loaded as shown in Fig. 9.

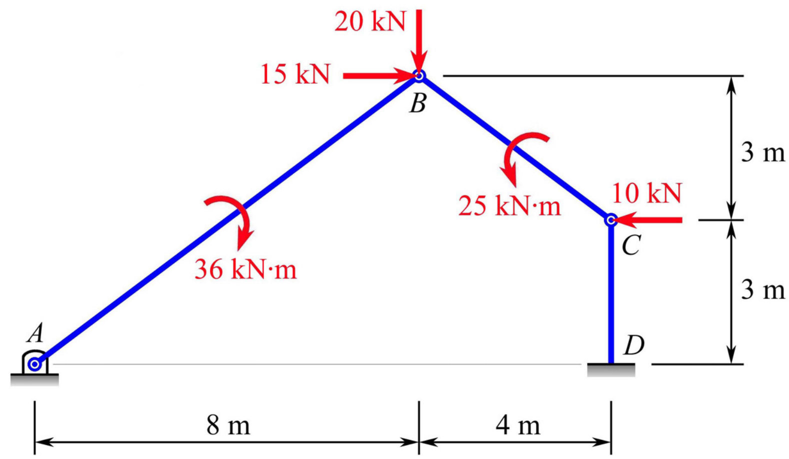

Fig. 9 A frame with hinge support at A and fixed support at D

Solution. We first draw the free-body diagram of the frame in Fig. 10.

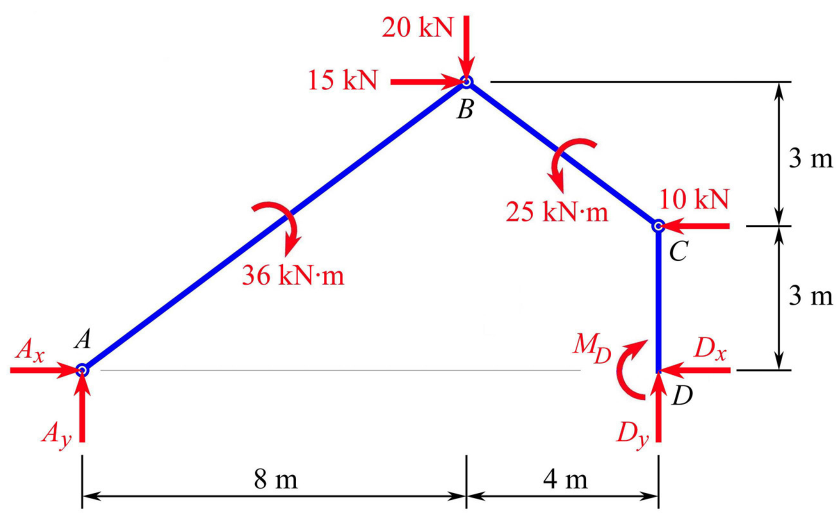

Fig. 10 Free-body diagram for the frame

Next, we draw a set of compatible virtual displacements for the frame as shown in Fig. 11. Note that we draw this virtual-displacement diagram with a *strategy* such that no unknowns *except* D_x will be involved in the total virtual work done. In Fig. 11, note the following:

(a) The displacement center of the member *AB* is at *A*.
(b) The displacement center of the member *BC* is at *E*.
(c) The displacement center of the member *CD* is at ∞.

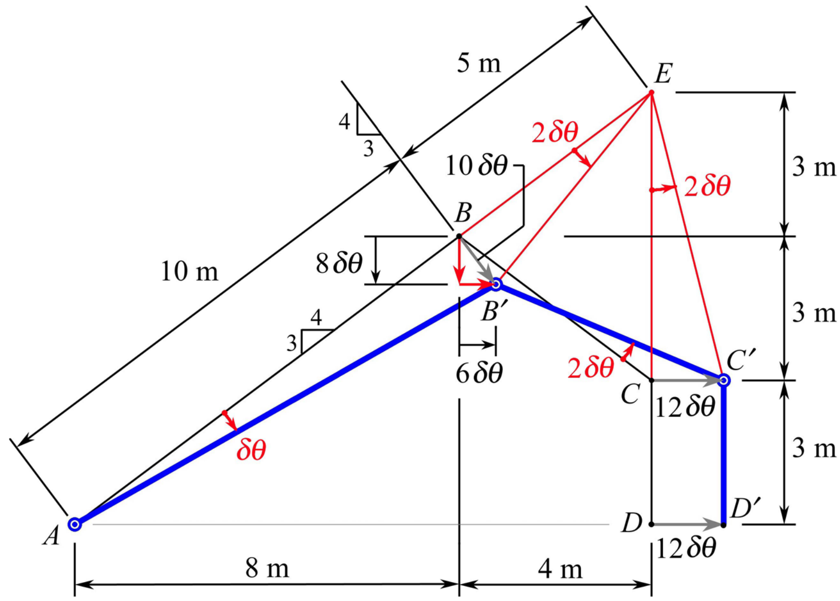

Fig. 11 Virtual-displacement diagram for use in solving for D_x

Referring to Figs. 10 and 11 and applying the virtual work principle, we write

$$\delta U = 0: \qquad 36\,(\delta\theta) + 15\,(6\,\delta\theta) + 20\,(8\,\delta\theta) + 25\,(2\,\delta\theta) + 10\,(-12\,\delta\theta) + D_x\,(-12\,\delta\theta) = 0$$

$$D_x = 18 \qquad\qquad \mathbf{D}_x = 18 \text{ kN} \leftarrow$$

Remarks. Notice that the free-body diagram for this frame contains **five** unknowns. The above solution by the *virtual work method* shows that we are able to solve for the designated unknown D_x without the need to solve simultaneous equations! This advantage may initially appear as a magic black box to students and cannot be matched by the solution using the conventional method.

IV. Concluding Remarks

The enthusiasm of an instructor about the beauty and powerfulness of the virtual work method can readily be contagious to the students. In teaching the virtual work method to students, it is important to refresh relevant fundamental concepts. The application of the concept of *displacement center* for each member in the system is what makes possible the use of just geometry and algebra (rather than differential calculus) as prerequisite mathematics for the teaching and learning of the *principle of virtual work* in statics.

Clearly, the advantages of the virtual work method lie in its conciseness in the principle, its visual elegance in the formulation of the solution via virtual displacement diagrams, and its tremendous saving in algebraic efforts by doing away with the need to solve tedious simultaneous equations in complex problems. These advantages are sparks that kindle students' interest in their learning of the virtual work method.

It is true that the drawing of the compatible virtual displacement for frames and machines involves considerable amount of basic geometry and requires good graphics skills. These aspects do present some degree of challenges to a number of beginning students. Nevertheless, the learning of the virtual work method is an excellent training ground for engineering and technology students to develop their visual skills in reading technical drawing and presenting technical conceptions.

References

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