From Conventional Method to Virtual Work Method in Statics: Three Major Steps and One Guiding Strategy

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Abstract

The transition in the learning of Statics from the traditional approach, which uses force and moment equilibrium equations, to the energy approach, which sets to zero the total virtual work done, involves learning a number of additional key concepts and a guiding strategy. Such a transition is often a challenge to students of Statics, which is a beginning fundamental course in most engineering curricula. It is presented in this paper *three major steps* and *one guiding strategy* for implementing the virtual work method. These steps and strategy have led to much better understanding and more effective learning of the virtual work method for students. The drawing of compatible virtual displacements for frames and machines involves basic geometry and requires good graphics skills. These aspects provide opportunities for students to reinforce their skills in geometry and graphics. Thus, the transformation of coverage and emphasis in Statics at Arkansas has resulted in equipping students with an added powerful analytical method and helped them enhance skills in reading drawings and presenting technical conceptions.

I. Introduction

In this knowledge age, the profession of engineering and the teaching of engineering are undergoing a transformation driven by new forces such as nano-sciences and new biology. At University of Arkansas, the traditional discipline of mechanical engineering now encompasses the new fields of MEMS and bio-nanotechnology, in addition to the traditional fields of thermal systems, mechanical design, mechanics, and materials. Moreover, the contents of several courses are transformed in coverage and emphases. In Statics, the teaching of the energy approach using virtual work method is being given significant emphasis soon after the teaching of the traditional approach using force and moment equilibrium equations.

Work is energy in transition to a system due to force or moment acting on the system during a displacement of the system. Work, as well as heat, is dependent on the path of a process. Like heat, work crosses the system boundary when the system undergoes a process. Unlike kinetic energy and potential energy, work is not a property possessed by a system. A *virtual displacement* of a body is a fictitious first-order differential displacement. A *virtual work* is the work done by force or moment during a virtual displacement of the system. The virtual work method has many applications, and it is often more powerful than the traditional method in solving problems involving frames or machines. By letting the free body of a system undergo a strategically chosen compatible virtual displacement in the virtual work method, we can solve for one specified unknown at a time in many complex as well as simple problems in Statics without having to solve coupled simultaneous equations. The virtual work method may initially come across as a magic black box to students, but it generally kindles great curiosity and interest in students of Statics. It is the aim of this paper to: (*a*) utilize displacement centers and just algebra and geometry (rather than differential calculus) as the prerequisite mathematics to compute virtual displacements, (*b*) present three major steps for implementing the virtual work method, and (*c*) propose a guiding strategy in choosing the virtual displacement for determining the specified unknown.

II. Fundamental Concepts

In mechanics, a body receives work from a force or a moment that acts on it if it undergoes a displacement in the direction of the force or moment, respectively, during the action. It is the force or moment, rather than the body, which does work. In teaching and learning the virtual work method, it is well to refresh the following fundamental concepts:

▪ Work of a force

If a force **F** acting on a body is *constant* and the displacement vector of the body from position A_1 to position A_2 during the action is **q**, then the *work* $U_{1\to 2}$ *of the force* **F** on the body is[1-4]

$$U_{1\to 2} = \mathbf{F}\cdot\mathbf{q} = Fq_{\parallel} \tag{1}$$

where F is the magnitude of **F** and $q_{\parallel}$ is the scalar component of **q** parallel to **F**. If the force is not constant, then integration may be used to compute the work of the force.

▪ Work of a moment

If a moment **M** (or a couple of moment **M**) acting on a body is *constant* and the angular displacement of the body from angular position θ_1 to angular position θ_2 in the direction of **M** during the action is $\Delta\theta$, then the *work* $U_{1\to 2}$ *of the moment* **M** on the body is[1-4]

$$U_{1\to 2} = M\,(\Delta\theta) \tag{2}$$

where M is the magnitude of **M**. If the moment is not constant, then integration may be used to compute the work of the moment.

▪ Rigid-body virtual displacement

In this paper, all bodies considered are rigid bodies or systems of pin-connected rigid bodies that can rotate frictionlessly at the pin joints. A *displacement* of a body is the change of position of the body. A *rigid-body displacement* of a body is the change of position of the body without inducing any strain in the body. A *virtual displacement* of a body is a fictitious first-order differential displacement, which is possible but does not actually take place. Furthermore, virtual displacements can be either *consistent* or *inconsistent* with constraints of the body.[1-5] A **rigid-body virtual displacement** of a body is a rigid-body displacement as well as a virtual displacement of the body, where the body undergoes a first-order differential deflection to a neighboring position, but the body experiences no axial strain at all, as illustrated in Fig. 1 for a body *AB* and in Fig. 2 for a hinged body *ABC*, which is composed of two rigid members *AB* and *BC* that are hinged together at *B*. (Notice that the hinged body *ABC* is a system of pin-connected rigid bodies.)

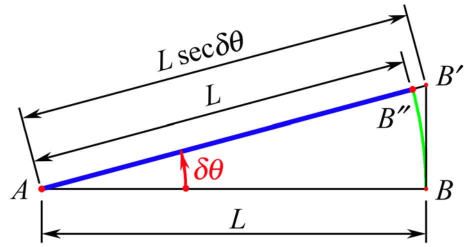

Fig. 1 Body *AB* undergoing a **rigid-body virtual displacement** to position *AB″*

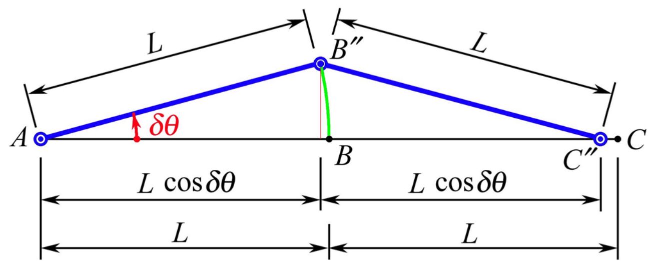

Fig. 2 Hinged body *ABC* undergoing a **rigid-body virtual displacement** to position *AB″C″*

■ Compatible virtual displacement

A **compatible virtual displacement** of a body is an imaginary *first-order* differential displacement, which conforms to the integrity (i.e., no breakage or rupture) of its free body within the framework of *first-order* differential change in geometry, where the body may be a particle, a rigid body, or a set of pin-connected rigid bodies.[4] A *compatible virtual displacement* of a body is *compatible with what is required in the virtual work method*; it is generally *different* from a rigid-body virtual displacement of the body. When a body is given a compatible virtual displacement, the body will undergo a first-order differential deflection to a neighboring position, and the body may experience, at most, a second-order (but *not* first-order) infinitesimal axial strain, as compared with a corresponding rigid-body virtual displacement of the same body. Recall that a *second-order* differential change in geometry is *a great deal smaller than* the *first-order* differential change in geometry and is negligible in the limit. This is illustrated in Fig. 3 for a single member *AB* and in Fig. 4 for a hinged body *ABC*.

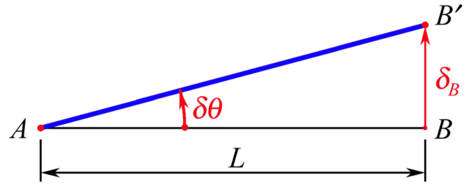

Fig. 3 Body *AB* undergoing a **compatible virtual displacement** to position *AB′*

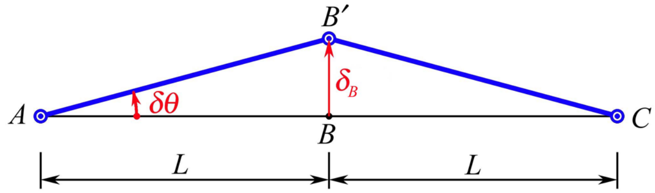

Fig. 4 Hinged body *ABC* undergoing a **compatible virtual displacement** to position *AB′C*

Using series expansion in terms of the *first-order* differential angular displacement $\delta\theta$, which is *infinitesimal*, we find that the distance between B'' and B' in Fig. 1 is

$$\overline{B''B'} = L\sec\delta\theta - L = L\left[1+\tfrac{1}{2}(\delta\theta)^2+\tfrac{5}{24}(\delta\theta)^4+\tfrac{61}{720}(\delta\theta)^6+\cdots\right]-L \approx \frac{L}{2}(\delta\theta)^2 \qquad (3)$$

Thus, the length $\overline{B''B'}$ is of the second order of $\delta\theta$ and is *negligible* in the virtual work method. The *compatible virtual displacement* of point B in Figs. 1, 3, and 4 is from B to B'. We find that

$$\overline{BB'} = \delta_B = L\tan\delta\theta = L\left[\delta\theta+\tfrac{1}{3}(\delta\theta)^3+\tfrac{2}{15}(\delta\theta)^5+\tfrac{17}{315}(\delta\theta)^7+\cdots\right]\approx L\,\delta\theta \qquad (4)$$

In Fig. 1, the lengths of the chord $\overline{BB'}$ and the arc $\overset{\frown}{BB''}$ can be taken as equal in the limit since the angle $\delta\theta$ is infinitesimally small. Therefore, the magnitude of the compatible linear virtual displacement of point *B*, as given by Eq. (4), may indeed be computed using the *radian measure formula* in calculus; i.e.,

$$s = r\theta \qquad (5)$$

where s is the arc subtending an angle θ in radian included by two radii of length r. In virtual work method, all *virtual displacements* can be *compatible virtual displacements*, and these two terms are understood to be interchangeable in the remainder of this paper.

■ Displacement center

Relations among the virtual displacements of certain points or members in a system can be found by using *differential calculus*, or the *displacement center*,[5] or both. The *displacement center* of a body is the point about which the body is perceived to rotate when it undergoes a virtual displacement. There are n displacement centers for a system composed of n pin-connected rigid bodies undergoing a set of virtual displacements; i.e., each member in such a system has its own displacement center. Generally, the **displacement center** of a body is located at the point of intersection of two straight lines that are drawn from two different points of the body *in the initial position* and are perpendicular to the virtual displacements of these two points, respectively.[5] This is illustrated in Fig. 5, where the body AB is imagined to slide on its supports to undergo a virtual displacement to the position $A'B'$, and its displacement center C is the point of intersection of the straight lines AC and BC that are drawn from points A and B and are perpendicular to their virtual displacements $\overrightarrow{AA'}$ and $\overrightarrow{BB'}$, respectively.

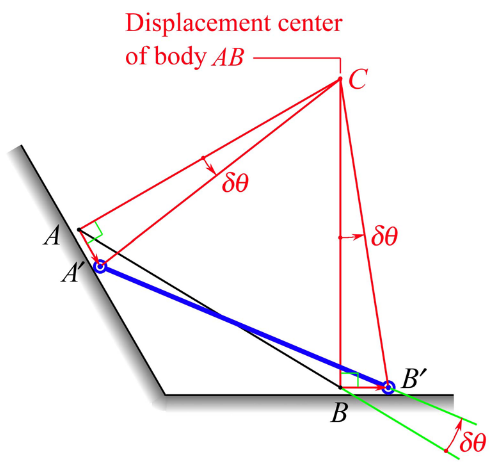

Fig. 5 Virtual displacement of body AB to position $A'B'$ with **displacement center** at C

It is often helpful to perceive the overall situation in Fig. 5 as an event where the body *AB* and its displacement center *C* form a "rigid triangular plate" that undergoes a rotation about *C* through an angle $\delta\theta$ from the initial position *ABC* to the new position $A'B'C$. In this event, all sides of this "rigid triangular plate" (i.e., the sides *AB*, *BC*, and *CA*), as well as any line that might be drawn on it, will and must rotate through the *same* angle $\delta\theta$, as indicated.

Sometimes, it is not necessary to use the procedure illustrated in Fig. 5 to locate the displacement center of a body. When a body undergoes a virtual displacement by simply rotating about a given point, then the **displacement center** of the body is simply located at the given point of rotation. This is illustrated in Figs. 6 and 7.

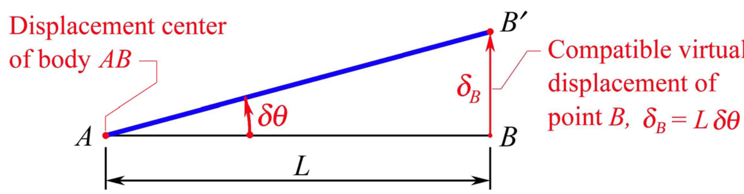

Fig. 6 Virtual displacement of body AB to position AB' with **displacement center** at *A*

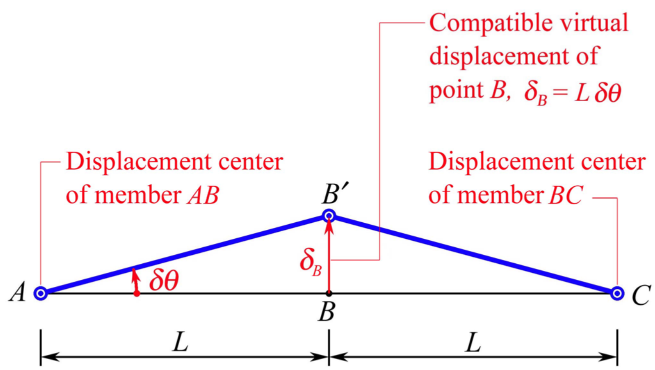

Fig. 7 Virtual displacement and the *two* **displacement centers** for the hinged body *ABC*

■ Principle of virtual work

Bodies considered here are rigid bodies or systems of pin-connected rigid bodies. The term "force system" denotes a system of *forces* and *moments*, if any. The work done by a force system on a body during a *virtual displacement* of the body is the **virtual work** of the force system. By Newton's third law, internal forces in a body, or a system of pin-connected rigid bodies, must occur in pairs; they are equal in magnitude and opposite in directions in each pair. Clearly, the total virtual work done by the internal forces during a virtual displacement of a body, or a system of pin-connected rigid bodies, must be zero. When a body, or a system of pin-connected rigid bodies, is in equilibrium, the resultant force and the resultant moment acting *on* its free body must both be zero.

The total virtual work done by the force system acting *on* the free body of a body is, by the distributive property of *dot product of vectors*, equal to the total virtual work done by the resultant force and the resultant moment acting on the free body, which are both zero if the body is in equilibrium. Therefore, we have the **principle of virtual work** in Statics, which may be stated as follows: *If a body is in equilibrium, the total virtual work of the external force system acting on its free body during any compatible virtual displacement of its free body is equal to zero, and conversely.* [1-4] Note that the body in this principle may be a particle, a set of connected particles, a rigid body, or a system of pin-connected rigid bodies (e.g., a frame or a machine). Using δU to denote the total virtual work done, we write the equation for this principle as

$$\delta U = 0 \tag{6}$$

■ **Conventional method** versus **virtual work method**

With the **conventional method**, equilibrium problems are solved by applying two basic equilibrium equations: (*a*) *force equilibrium equation*, and (*b*) *moment equilibrium equation*; i.e.,

$$\Sigma \mathbf{F} = \mathbf{0} \tag{7}$$

$$\Sigma \mathbf{M}_P = \mathbf{0} \tag{8}$$

With the **virtual work method**, equilibrium problems are solved by applying the *virtual work equation*, which sets to zero the total virtual work δU done by the force system on the free body during a chosen *compatible virtual displacement* of the free body; i.e.,

$$\delta U = 0 \qquad \text{(Repeated)} \tag{6}$$

III. Three Major Steps and One Guiding Strategy in Virtual Work Method: Examples

There are **three major steps** in using the *virtual work method*. **Step 1:** *Draw the free-body diagram (FBD)*. **Step 2:** *Draw the virtual-displacement diagram* (*VDD*) with a guiding strategy. **Step 3:** *Set to zero the total virtual work done*. The **guiding strategy** in step 2 is to give the free body a compatible virtual displacement in such a way that the *one* specified unknown, but *no other unknowns*, will be involved in the virtual work done. That is it: three major steps and one guiding strategy in the virtual work method! This is demonstrated in the following examples.

Example 1. Determine the vertical reaction force $\mathbf{B}_y$ at the roller support B of the simple beam loaded as shown in Fig. 8 by using (*a*) the *conventional method*, and (*b*) the *virtual work method*.

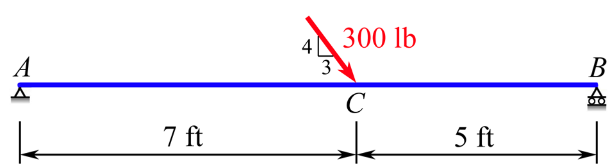

Fig. 8 A simple beam carrying an inclined concentrated load

Solution. Note that color codes are here employed in the solution to enhance head-to-head comparison of (*a*) the *conventional method*, and (*b*) the *virtual work method*.

(*a*) **Conventional method to solve for $\mathbf{B}_y$:** We first draw the free-body diagram shown in Fig. 9, where we have replaced the 300-lb force at C with its rectangular components.

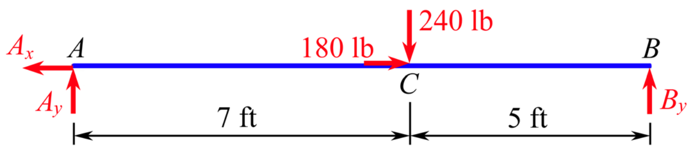

Fig. 9 Free-body diagram for the beam

In this method, we refer to Fig. 9 and apply Eq. (8) to write

$+\circlearrowleft \Sigma M_A = 0$: $\quad -7(240) + 12B_y = 0 \quad \therefore \quad B_y = 140 \quad$ $\boxed{\mathbf{B}_y = 140 \text{ lb} \uparrow}$

(*b*) **Virtual work method to solve for $\mathbf{B}_y$**: For step 1 in the solution, we draw the *FBD* for the beam as shown in Fig. 9.

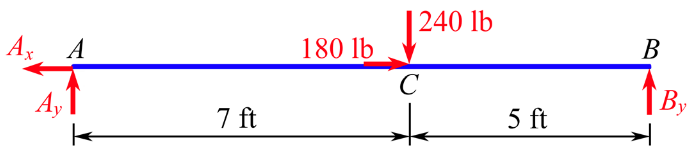

Fig. 9 Free-body diagram for the beam (Repeated)

For step 2 in the solution, we keep an eye on the *FBD* in Fig. 9 and draw the *VDD* with a **strategy** for the beam as shown in Fig. 10, where we rotate the beam counterclockwise through an angular displacement $\delta\theta$ about point *A*. The resulting *VDD* will involve the unknown B_y, but *no other unknowns*, in the total virtual work done.

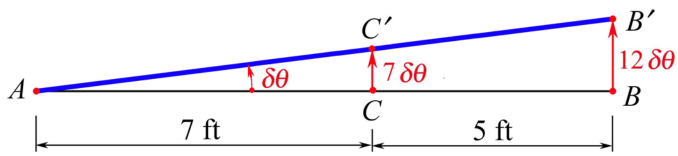

Fig. 10 Virtual-displacement diagram to involve B_y in $\delta U = 0$ (displ. ctr. at *A*)

For step 3 in the solution, we refer to Figs. 9 and 10 and apply Eqs. (1) and (6) to write

$\delta U = 0$: $\quad 240\,(-7\,\delta\theta) + B_y\,(12\delta\theta) = 0 \quad \therefore \quad B_y = 140 \quad$ $\boxed{\mathbf{B}_y = 140 \text{ lb} \uparrow}$

Remark. We see in Example 1 that what the conventional method can solve, the virtual work method can similarly solve, too. However, virtual work method is best used to solve more complex, rather than simple, problems.

Example 2. Using virtual work method, determine the vertical reaction force $\mathbf{A}_y$ at the fixed support *A* of the combined beam (called a *Gerber beam*) loaded as shown in Fig. 11.

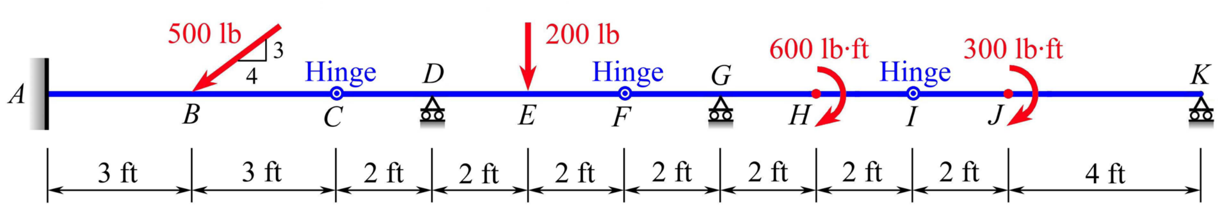

Fig. 11 A combined beam with hinge connections at *C*, *F*, and *I*

Solution. For step 1 in the solution, we draw the *FBD* for the beam as shown in Fig. 12.

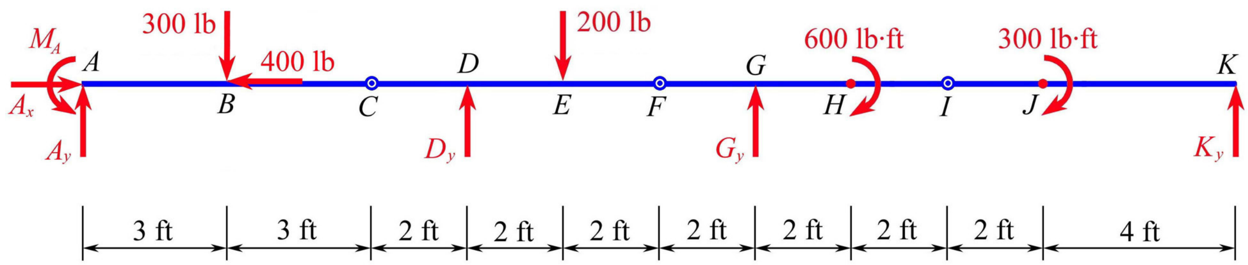

Fig. 12 Free-body diagram for the combined beam

For step 2 in the solution, we keep an eye on the *FBD* in Fig. 12 and draw the *VDD* with a **strategy** for the beam as shown in Fig. 13, where segments *ABC*, *CDEF*, *FGHI*, and *IJK* have displacement centers located at ∞, and at points *D*, *G*, and *K*, respectively. The resulting *VDD* will involve no unknowns *except* A_y in the total virtual work done.

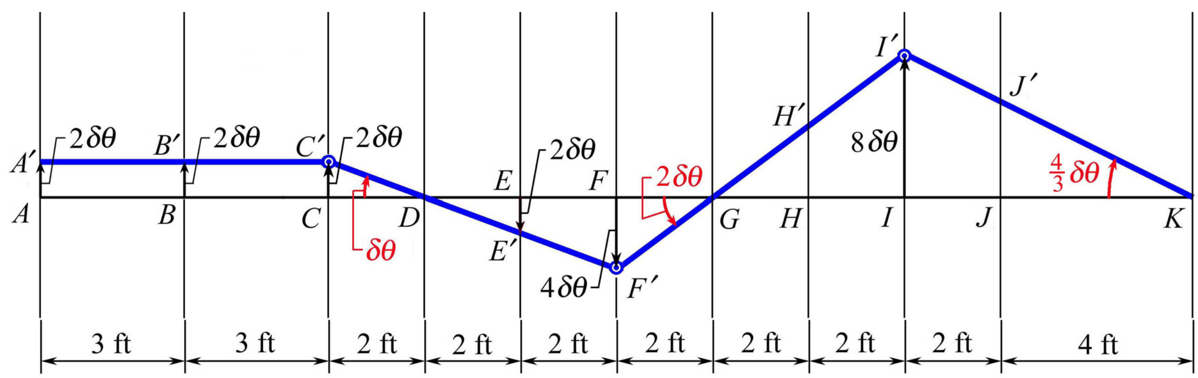

Fig. 13 Virtual-displacement diagram for the combined beam to involve A_y in $\delta U = 0$

For step 3 in the solution, we refer to Figs. 12 and 13 and apply Eqs. (1), (2), and (6) to write

$$\delta U = 0: \qquad A_y(2\,\delta\theta) + 300(-2\,\delta\theta) + 200(2\,\delta\theta) + 600(-2\,\delta\theta) + 300\left(\tfrac{4}{3}\delta\theta\right) = 0$$

$$A_y = 500 \qquad\qquad \mathbf{A}_y = 500 \text{ lb} \uparrow$$

Remarks. With the **conventional method**, we have to refer to the *FBD* and write

- At hinge *I*, $M_I = 0$: $\quad 6K_y - 300 = 0 \quad$ (1)
- At hinge *F*, $M_F = 0$: $\quad 12K_y - 300 - 600 + 2G_y = 0 \quad$ (2)
- At hinge *C*, $M_C = 0$: $\quad 18K_y - 300 - 600 + 8G_y - 4(200) + 2D_y = 0 \quad$ (3)
- For the entire beam, $+\uparrow \Sigma F_y = 0$: $\quad A_y + D_y + G_y + K_y - 300 - 200 = 0 \quad$ (4)

These four simultaneous equations yield: $K_y = 50$, $G_y = 150$, $D_y = -200$, and $A_y = 500$. Thus, the **conventional method** eventually yields the same solution: $\mathbf{A}_y = 500 \text{ lb} \uparrow$

Example 3. Using virtual work method, determine the reaction moment $\mathbf{M}_D$ at the fixed support D of the frame loaded as shown in Fig. 14.

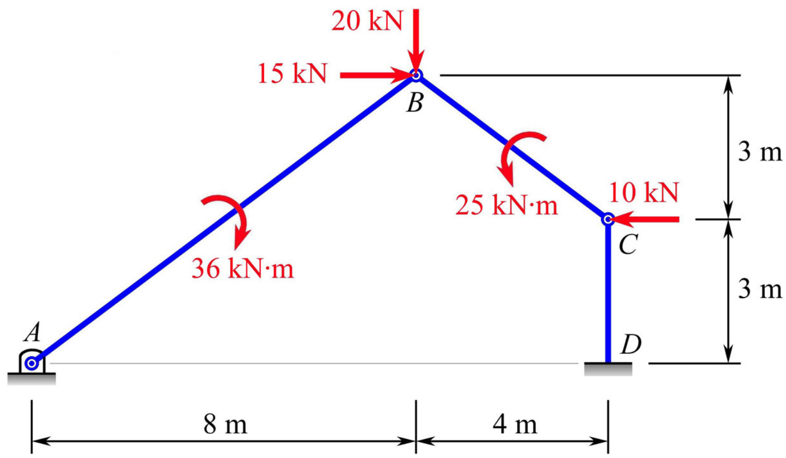

Fig. 14 A frame with hinge support at A and fixed support at D

Solution. For step 1 in the solution, we draw the *FBD* for the frame as shown in Fig. 15.

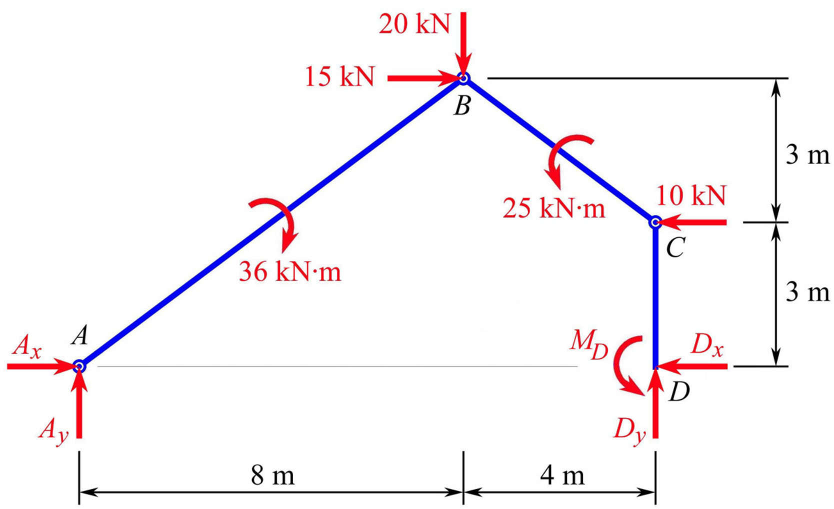

Fig. 15 Free-body diagram for the frame

For step 2 in the solution, we keep an eye on the *FBD* in Fig. 15 and draw the *VDD* with a **strategy** for the frame as shown in Fig. 16, where we let the angular virtual displacement of member *AB* be $\delta\theta$, and members *AB*, *BC*, and *CD* have displacement centers located at points *A*, *E*, and *D*, respectively. The resulting *VDD* will involve no unknowns *except* M_D in the total virtual work done. In Fig. 16, pay special attention to the following:

- The compatible virtual displacement $\overrightarrow{BB'}$ of point B is such that $\overline{BB'} \perp \overline{AB}$ and $\overline{BB'} = 10\,\delta\theta$.
- Each of the three sides (i.e., *BC*, *CE*, and *EB*) of the "rigid triangular plate" *BCE* rotates counterclockwise through the same angle given by

$$\overline{BB'}\,/\,\overline{BE} = (10\,\delta\theta)\,/\,5 = 2\,\delta\theta$$

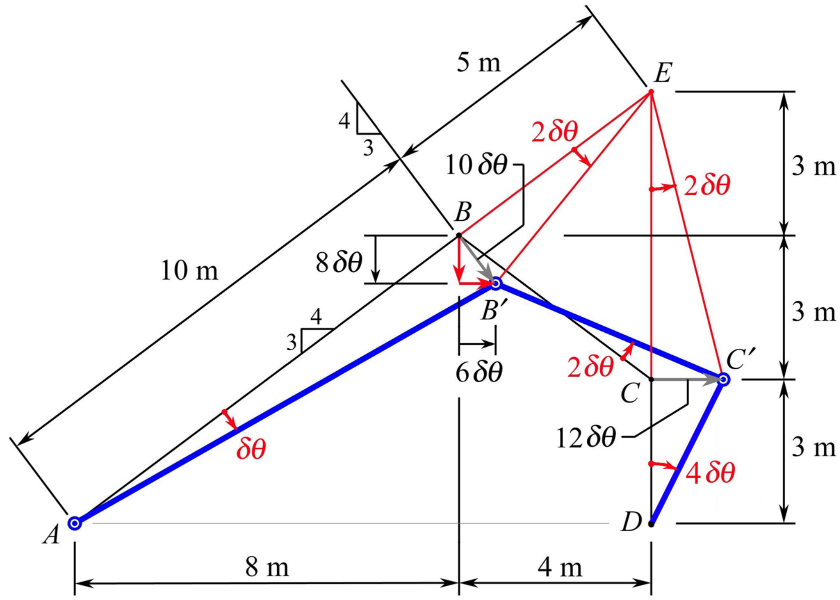

Fig. 16 Virtual-displacement diagram to involve M_D in the virtual work done

For step 3 in the solution, we refer to Figs. 15 and 16 and apply Eqs. (1), (2), and (6) to write

$$\delta U = 0: \qquad 36(\delta\theta) + 15(6\,\delta\theta) + 20(8\,\delta\theta) + 25(2\,\delta\theta) + 10(-12\,\delta\theta) + M_D(-4\,\delta\theta) = 0$$

$$M_D = 54 \qquad\qquad \mathbf{M}_D = 54\ \text{kN·m}\ \circlearrowleft$$

Remarks. With the **conventional method**, we have to refer to the free-body diagram of the frame in Fig. 15 and write

- At hinge C, $M_C = 0$: $\qquad M_D - 3D_x = 0 \qquad (1)$
- At hinge B, $M_B = 0$: $\qquad M_D - 6D_x + 4D_y - 3(10) + 25 = 0 \qquad (2)$
- For the entire frame, $+\circlearrowleft \Sigma M_A = 0$:

$$M_D + 12D_y + 3(10) + 25 - 6(15) - 8(20) - 36 = 0 \qquad (3)$$

These three simultaneous equations yield: $M_D = 54$, $D_x = 18$, and $D_y = 14.75$. Thus, the **conventional method** eventually yields the same solution: $\mathbf{M}_D = 54\ \text{kN·m}\ \circlearrowleft$

IV. Concluding Remarks

Solutions for *simple* equilibrium problems by the virtual work method may come across as "unconventional" when compared with those by the conventional method, as illustrated in Example 1 in Section III. Well, Example 1 was provided merely as a teaching and learning example to bring out contrasts between the conventional method and the virtual work method. After all, the virtual work method has been shown as a fabulous and powerful method in solving decently challenging problems as illustrated in Examples 2 and 3.

The implementation of the proposed **three major steps** and **one guiding strategy** in the virtual work method, as described and illustrated in Section III, has greatly helped students understand and implement the virtual work method, which is covered in one and half weeks in each semester at University of Arkansas in the past several years. Thus, the transition in the learning of Statics from the traditional approach to the energy approach, which is emphasized in recent years, has been made smoother and easier. The enthusiasm of an instructor about the beauty and powerfulness of the virtual work method can readily be contagious to the students. The application of the concept of *displacement center* for each member in a system is what makes possible the use of just algebra and geometry (rather than differential calculus) as prerequisite mathematics for the teaching and learning of the *principle of virtual work* in Statics.

Clearly, the advantages of the virtual work method lie in its conciseness in the principle, its visual elegance in the formulation of the solution via virtual-displacement diagrams, and its saving in algebraic effort by doing away with the need to solve simultaneous equations in complex problems. The virtual work method may initially appear as a magic black box to students, but the advantages and elegance witnessed by students are sparks that kindle their interest in learning the virtual work method in particular and the subject of Statics in general.

It is true that the drawing of compatible virtual displacements for frames and machines involves basic geometry and requires good graphics skills. These aspects do present some degree of challenges to a number of beginning students. Nevertheless, the learning of the virtual work method is an excellent training ground for engineering and technology students to develop their visual skills in reading technical drawings and presenting technical conceptions.

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