

## Teaching Students Work and Virtual Work Method in Statics: A Guiding Strategy with Illustrative Examples

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### Abstract

A *virtual displacement* is an imaginary differential displacement that may *not* really take place. A virtual displacement may be *either consistent* with constraints at supports *or inconsistent* with constraints at supports. A **virtual work** is the work done by force or moment during a virtual displacement of the system. The **virtual work method** can be applied to solve problems involving *either machines* (structures with movable members) *or frames* (structures with *no* movable members). By letting the free body of a system undergo a strategically chosen **compatible virtual displacement** in the virtual work method, we can *solve for one specified unknown at a time* in many complex as well as simple problems in mechanics without having to solve coupled simultaneous equations. The virtual work method may initially appear as a magic black box to students, but it generally kindles great curiosity and interest in students of statics. This paper proposes an approach consisting of *three major steps* and *one guiding strategy* for implementing the virtual work method. It results in great learning of the virtual work method for students.

### I. Introduction

**Work** is energy in transition to a system due to force or moment acting on the system through a displacement of the system, while *heat* is energy in transition to a system due to temperature difference between the system and its surroundings. Work, as well as heat, is dependent on the path of a process. Like heat, work crosses the system boundary when the system undergoes a process. Unlike kinetic energy and potential energy, work is not a property possessed by a system. Many textbooks in statics show the use of virtual work method to solve problems involving *mainly machines*, where the virtual displacements are usually chosen to be **consistent** with constraints at supports. The virtual work method *can equally be used* to solve problems involving **frames** in statics. Readers may refer to textbooks by Beer and Johnston,<sup>1-2</sup> Huang,<sup>3</sup> Jong and Rogers,<sup>4</sup> etc., where virtual displacements **inconsistent** with constraints at supports are strategically chosen to solve equilibrium problems of *frames*, which are fully constrained at supports.

This paper is aimed at doing the following: (a) sharpen the concept of *work* for students, (b) compare head to head the *virtual work method* with the *conventional method* using an example, (c) use **displacement center**<sup>5</sup> and just algebra and geometry as the prerequisite mathematics to compute virtual displacements, (d) propose **three major steps** in the virtual work method, (e) propose a **guiding strategy** for choosing the virtual displacement that is the best for solving one specified unknown, and (f) demonstrate (in **Appendix A**) the evidence that the conventional method (*without* displacement center) requires using **differential calculus** in determining a certain virtual displacement. For benefits of a wider range of readers having varying familiarity with the subject, this paper contains illustrative examples with different levels of complexity.

## II. Fundamental Concepts

Engineering students learn the definition of *work* when they take the course in physics usually in their freshman year. In mechanics, a body receives work from a force or a moment that acts on it if it undergoes a displacement in the direction of the force or moment, respectively, during the action. It is the force or moment, rather than the body, which does work. In teaching and learning the virtual work method, it is well to refresh the following fundamental concepts:

### ■ Work of a force

The **work**  $U_{1 \rightarrow 2}$  done by a force  $\mathbf{F}$  on a body moving from position  $A_1$  along a path  $C$  to position  $A_2$  is defined by a line integral. It is given by<sup>1-4, 6-7</sup>

$$U_{1 \rightarrow 2} = \int_{A_1}^{A_2} \mathbf{F} \cdot d\mathbf{r} \quad (1)$$

where  $\cdot$  denotes a dot product, and  $d\mathbf{r}$  is the differential displacement of the body moving along the path  $C$  during the action of  $\mathbf{F}$  on the body. If the force  $\mathbf{F}$  is *constant* and the displacement vector of the body during the action is  $\mathbf{q}$ , then the work done on the body is given by

$$U_{1 \rightarrow 2} = \mathbf{F} \cdot \mathbf{q} = Fq_{\parallel} \quad (2)$$

where  $F$  is the magnitude of  $\mathbf{F}$  and  $q_{\parallel}$  is the scalar component of  $\mathbf{q}$  parallel to  $\mathbf{F}$ . If we let the angle between the positive directions of  $\mathbf{F}$  and  $\mathbf{q}$  be  $\phi$  and assume that both  $\mathbf{F}$  and  $\mathbf{q}$  are not zero, then the values of both  $q_{\parallel}$  and  $U_{1 \rightarrow 2}$  are negative if and only if  $90^\circ < \phi \leq 180^\circ$ .

### ■ Work of a moment

The **work**  $U_{1 \rightarrow 2}$  done by a moment  $\mathbf{M}$  (or a couple of moment  $\mathbf{M}$ ) on a body during its finite rotation, parallel to  $\mathbf{M}$ , from angular position  $\theta_1$  to angular position  $\theta_2$  is given by<sup>1-4, 6-7</sup>

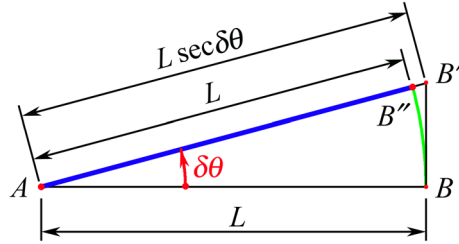
$$U_{1 \rightarrow 2} = \int_{\theta_1}^{\theta_2} M d\theta \quad (3)$$

If the moment  $\mathbf{M}$  is *constant* and the angular displacement of the body in the direction of  $\mathbf{M}$  during the action is  $\Delta\theta$ , then the work done on the body is given by

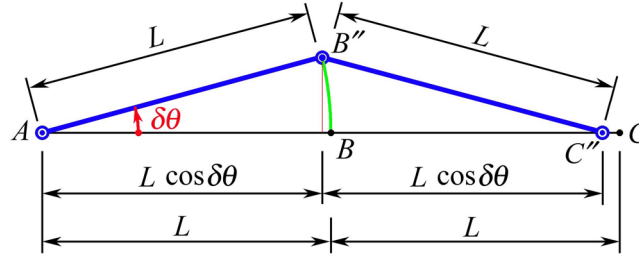
$$U_{1 \rightarrow 2} = M (\Delta\theta) \quad (4)$$

### ■ Compatible virtual displacement *versus* rigid-body virtual displacement

In this paper, all bodies considered are rigid bodies or systems of pin-connected rigid bodies that can rotate frictionlessly at the pin joints. A **displacement** of a body is the change of position of the body. A **rigid-body displacement** of a body is the change of position of the body without inducing any strain in the body. A **virtual displacement** of a body, or a system of pin-connected rigid bodies, is an imaginary differential displacement, which is *infinitesimal* and is possible but may not really take place. **First**, we note that a **rigid-body virtual displacement** of a body is an imaginary differential rigid-body displacement of the body, which is *infinitesimal* and *exact*, as illustrated in Fig. 1 for a single member  $AB$  and in Fig. 2 for a hinged beam  $ABC$ . (Notice that the hinged beam  $ABC$  is a system of pin-connected rigid bodies.)

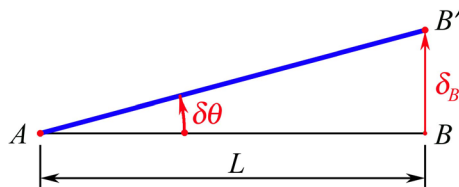


**Fig. 1** Body  $AB$  undergoing a **rigid-body virtual displacement** to position  $AB''$

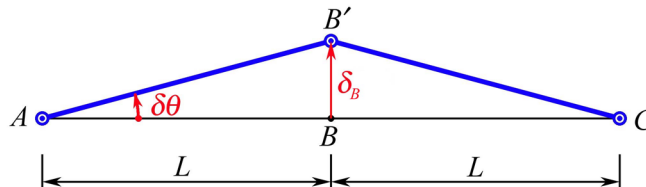


**Fig. 2** Hinged beam  $ABC$  undergoing a **rigid-body virtual displacement** to position  $AB''C''$

Now, a **compatible virtual displacement** of a body is an imaginary *first-order* differential displacement, which conforms to the integrity (i.e., no breakage or rupture) of its free body within the framework of *first-order* differential change in geometry, where the body may be a particle, a rigid body, or a set of pin-connected rigid bodies. A *compatible virtual displacement* of a body is *compatible with what is required* in the virtual work method; it is generally *different* from a rigid-body virtual displacement of the body. As in calculus, a *second-order* differential change in geometry is *a great deal smaller* than the *first-order* differential change in geometry, since all differential changes are infinitesimal. It is well to note that a **compatible virtual displacement** of a body may have, at most, a **second-order** (but *not* first-order) **infinitesimal differential change in its geometry** as compared with a corresponding rigid-body virtual displacement of the same body. This is illustrated in Fig. 3 for a single member  $AB$  and in Fig. 4 for a hinged beam  $ABC$ .



**Fig. 3** Body  $AB$  undergoing a **compatible virtual displacement** to position  $AB'$



**Fig. 4** Hinged beam  $ABC$  undergoing a **compatible virtual displacement** to position  $AB'C$

Using series expansion in terms of the *first-order* differential angular displacement  $\delta\theta$ , which is *infinitesimal*, we find that the distance between  $B''$  and  $B'$  in Fig. 1 is

$$\overline{B''B'} = L \sec \delta\theta - L = L \left[ 1 + \frac{1}{2}(\delta\theta)^2 + \frac{5}{24}(\delta\theta)^4 + \frac{61}{720}(\delta\theta)^6 + \dots \right] - L \approx \frac{L}{2}(\delta\theta)^2 \quad (5)$$

Thus, the length  $\overline{B''B'}$  is of the second order of  $\delta\theta$  and is negligible in the virtual work method. The *compatible virtual displacement* of point  $B$  in Figs. 1, 3, and 4 is from  $B$  to  $B'$ . We find that

$$\overline{BB'} = \delta_B = L \tan \delta\theta = L \left[ \delta\theta + \frac{1}{3}(\delta\theta)^3 + \frac{2}{15}(\delta\theta)^5 + \frac{17}{315}(\delta\theta)^7 + \dots \right] \approx L \delta\theta \quad (6)$$

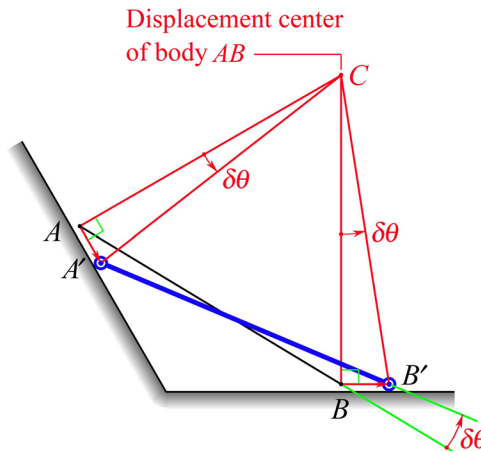
In Fig. 1, the lengths of the chord  $\overline{BB'}$  and the arc  $\widehat{BB''}$  can be taken as equal in the limit since the angle  $\delta\theta$  is infinitesimally small. Therefore, the magnitude of the compatible linear virtual displacement of point  $B$ , as given by Eq. (6), may indeed be computed using the *radian measure formula* in calculus; i.e.,

$$s = r\theta \quad (7)$$

where  $s$  is the arc subtending an angle  $\theta$  in radian included by two radii of length  $r$ . In virtual work method, all *virtual displacements* are meant to be *compatible virtual displacements*, and these two terms are understood to be interchangeable in the remainder of this paper.

### ■ Displacement center

Relations among the virtual displacements of certain points or members in a system can be found by using *differential calculus*, or the *displacement center*,<sup>5</sup> or both. The *displacement center* of a body is the point about which the body is perceived to rotate when it undergoes a virtual displacement. There are  $n$  displacement centers for a system composed of  $n$  pin-connected rigid bodies undergoing a set of virtual displacements; i.e., each member in such a system has its own displacement center. Generally, the **displacement center** of a body is located at the point of intersection of two straight lines that are drawn from two different points of the body *in the initial position* and are perpendicular to the virtual displacements of these two points, respectively. Readers familiar with dynamics would be correct to infer that the displacement center corresponded to the **velocity center**<sup>4</sup> of the body if the *virtual displacements* were the *velocities* of those two points on the body. This is illustrated in Fig. 5, where the body  $AB$  is imagined to slide on its supports to undergo a virtual displacement to the position  $A'B'$ , and its displacement center  $C$  is the point of intersection of the straight lines  $AC$  and  $BC$  that are drawn from points  $A$  and  $B$  and are perpendicular to their virtual displacements  $\overline{AA'}$  and  $\overline{BB'}$ , respectively.

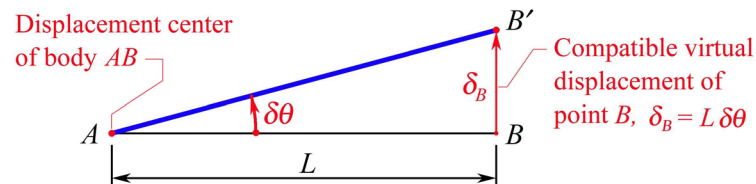


**Fig. 5** Virtual displacement of body  $AB$  to position  $A'B'$  with **displacement center** at  $C$

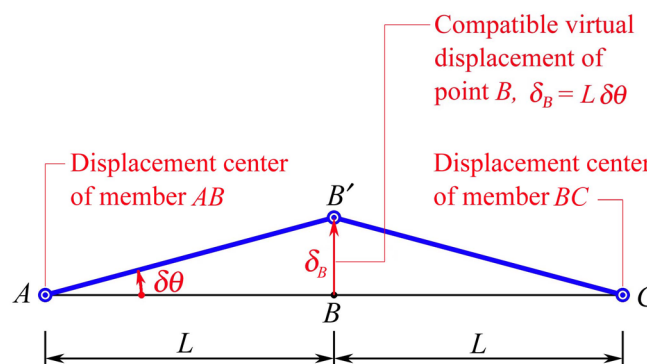


Students will find it helpful to perceive the overall situation in Fig. 5 as an event where the body  $AB$  and its displacement center  $C$  form a “rigid triangular plate” that undergoes a rotation about  $C$  through an angle  $\delta\theta$  from the initial position  $ABC$  to the new position  $A'B'C$ . In this event, all sides of this “rigid triangular plate” (i.e., the sides  $AB$ ,  $BC$ , and  $CA$ ), as well as any line that might be drawn on it, will and must rotate through the *same* angle  $\delta\theta$ , as indicated.

Sometimes, it is not necessary to use the procedure illustrated in Fig. 5 to locate the displacement center of a body. When a body undergoes a virtual displacement by simply rotating about a given point, then the **displacement center** of the body is simply located at the given point of rotation. This is illustrated in Figs. 6 and 7.



**Fig. 6** Virtual displacement of body  $AB$  to position  $AB'$  with **displacement center** at  $A$



**Fig. 7** Virtual displacement and the *two* **displacement centers** for the hinged system  $ABC$

Additional examples may be found later in this paper. Nonetheless, we attribute to the concept of **displacement center**,<sup>5</sup> which makes possible the use of just algebra and geometry (rather than differential calculus, as illustrated in **Appendix A**) as prerequisite mathematics for, and allowed the great expansion of, the use of the *principle of virtual work* in **statics**<sup>4</sup> and the *principle of virtual work in kinetics* as well as the *principle of generalized virtual work* in **Dynamics**.<sup>4</sup>

## ■ Principle of virtual work

Historical studies show that on February 26, 1715, the Swiss mathematician Johann Bernoulli (1667-1748) communicated to Pierre Varignon (1654-1722) the principle of virtual velocities in analytical form for the first time. That was the forerunner of the principle of virtual work today. The approach to mechanics based on the principle of virtual work was formally treated by Joseph Louis Lagrange (1736-1813) in his *Mécanique Analytique* published in 1788.

Keep in mind that bodies considered here are rigid bodies. The term “force system” denotes a system of *forces* and *moments*, if any. The work done by a force system on a body during a *vir-*

*tual displacement* of the body is the **virtual work** of the force system. By Newton's third law, internal forces in a body, or a system of pin-connected rigid bodies, must occur in pairs; they are equal in magnitude and opposite in directions in each pair. Clearly, the total virtual work done by the internal forces during a virtual displacement of a body, or a system of pin-connected rigid bodies, must be zero. When a body, or a system of pin-connected rigid bodies, is in equilibrium, the resultant force and the resultant moment acting *on* its free body must both be zero.

The total virtual work done by the force system acting *on* the free body of a body is, by the distributive property of *dot product of vectors*, equal to the total virtual work done by the resultant force and the resultant moment acting on the free body, which are both zero if the body is in equilibrium. Therefore, we have the **principle of virtual work** in statics, which may be stated as follows: *If a body is in equilibrium, the total virtual work of the external force system acting on its free body during any compatible virtual displacement of its free body is equal to zero, and conversely*. Note that the body in this principle may be a particle, a set of connected particles, a rigid body, or a system of pin-connected rigid bodies (e.g., a frame or a machine). Using  $\delta U$  to denote the total virtual work done, we write the equation for this principle as

$$\delta U = 0 \quad (8)$$

### III. Conventional Method versus Virtual Work Method: Example

With the **conventional method**, equilibrium problems are solved by applying two basic equilibrium equations: (a) *force equilibrium equation*, and (b) *moment equilibrium equation*; i.e.,

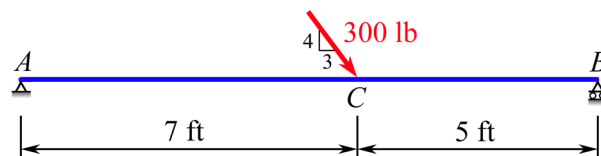
$$\Sigma \mathbf{F} = \mathbf{0} \quad (9)$$

$$\Sigma \mathbf{M}_p = \mathbf{0} \quad (10)$$

With the **virtual work method**, equilibrium problems are solved by applying the *virtual work equation*, which sets to zero the total virtual work  $\delta U$  done by the force system on the free body during a chosen *compatible virtual displacement* of the free body; i.e.,

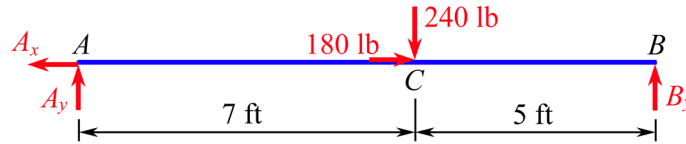
$$\delta U = 0 \quad (\text{Repeated}) \quad (8)$$

**Example 1.** Determine the reactions at supports *A* and *B* of the simple beam loaded as shown in Fig. 8 by using (a) the *conventional method*, and (b) the *virtual work method*. [Note that color codes are employed to enhance head-to-head comparison of **method (a)** with **method (b)**.]



**Fig. 8** A simple beam carrying an inclined concentrated load

- **Conventional method to solve for  $A_x$ :** We first draw the free-body diagram shown in Fig. 9, where we have replaced the 300-lb force at *C* with its rectangular components.



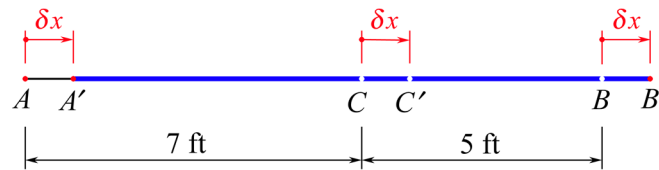
**Fig. 9** Free-body diagram for the beam

In this method, we refer to Fig. 9 and apply Eq. (9) to write

$$\pm \rightarrow \sum F_x = 0: \quad -A_x + 180 = 0 \quad \therefore A_x = 180$$

$$\mathbf{A}_x = 180 \text{ lb } \leftarrow$$

- **Virtual work method to solve for  $A_x$ :** We give the beam a *virtual displacement* shown in Fig. 10, which will *strategically* involve  $A_x$ , but *no other unknowns*, in the virtual work done.



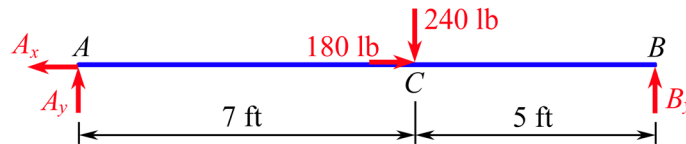
**Fig. 10** Virtual-displacement diagram to involve  $A_x$  in the virtual work done (displ. ctr. at  $\infty$ )

In this method, we refer to Figs. 9 and 10 and apply Eqs. (2) and (8) to write

$$\delta U = 0: \quad A_x (-\delta x) + 180 (\delta x) = 0 \quad \therefore A_x = 180$$

$$\mathbf{A}_x = 180 \text{ lb } \leftarrow$$

- **Conventional method to solve for  $A_y$ :** For ease of reference, we repeat Fig. 9 as follows:



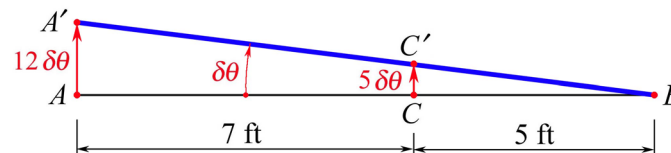
**Fig. 9** Free-body diagram for the beam (repeated)

In this method, we refer to Fig. 9 and apply Eq. (10) to write

$$+\circlearrowleft \sum M_B = 0: \quad -12A_y + 5(240) = 0 \quad \therefore A_y = 100$$

$$\mathbf{A}_y = 100 \text{ lb } \uparrow$$

- **Virtual work method to solve for  $A_y$ :** We give the beam a *virtual displacement* shown in Fig. 11, which will *strategically* involve  $A_y$ , but *no other unknowns*, in the virtual work done.



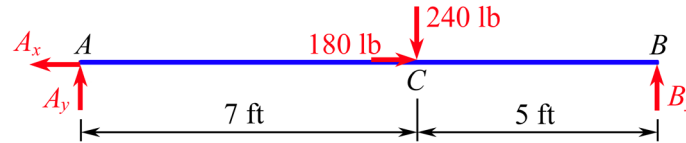
**Fig. 11** Virtual-displacement diagram to involve  $A_y$  in the virtual work done (displ. ctr. at B)

In this method, we refer to Figs. 9 and 11 and apply Eqs. (2) and (8) to write

$$\delta U = 0: \quad A_y (12\delta\theta) + 240 (-5\delta\theta) = 0 \quad \therefore A_y = 100$$

$$\mathbf{A}_y = 100 \text{ lb } \uparrow$$

- **Conventional method to solve for  $B_y$**  : For ease of reference, we repeat Fig. 9 as follows:



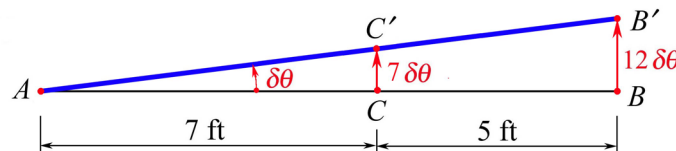
**Fig. 9** Free-body diagram for the beam (repeated)

In this method, we refer to Fig. 9 and apply Eq. (10) to write

$$+\circlearrowleft \sum M_A = 0: \quad -7(240) + 12B_y = 0 \quad \therefore B_y = 140$$

$$B_y = 140 \text{ lb } \uparrow$$

- **Virtual work method to solve for  $B_y$**  : We give the beam a *virtual displacement* shown in Fig. 12, which will *strategically* involve  $B_y$ , but *no other unknowns*, in the virtual work done.



**Fig. 12** Virtual-displacement diagram to involve  $B_y$  in the virtual work done (displ. ctr. at A)

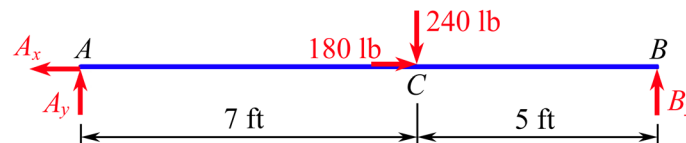
In this method, we refer to Figs. 9 and 12 and apply Eqs. (2) and (8) to write

$$\delta U = 0: \quad 240(-7 \delta \theta) + B_y(12 \delta \theta) = 0 \quad \therefore B_y = 140$$

$$B_y = 140 \text{ lb } \uparrow$$

**Remark.** Once we have determined that  $A_y = 100$  lb, we may make use of this solution to determine the value of the unknown reaction  $B_y$  in *alternative* ways as follows:

- **Conventional method to solve for  $B_y$**  : For ease of reference, we repeat Fig. 9 as follows:



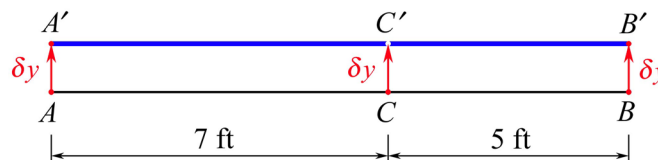
**Fig. 9** Free-body diagram for the beam (repeated)

In this method, we refer to Fig. 9 and apply Eq. (9) to write

$$+\uparrow \sum F_y = 0: \quad A_y - 240 + B_y = 0 \quad \therefore B_y = 240 - A_y = 140$$

$$B_y = 140 \text{ lb } \uparrow$$

- **Virtual work method to solve for  $B_y$**  : We give the beam a *virtual displacement* shown in Fig. 13, which will involve  $B_y$  and  $A_y$  in the virtual work done.



**Fig. 13** Virtual-displacement diagram to involve  $B_y$  &  $A_y$  in the virtual work done (displ. ctr. at  $\infty$ )

In this method, we refer to Figs. 9 and 13 and apply Eqs. (2) and (8) to write

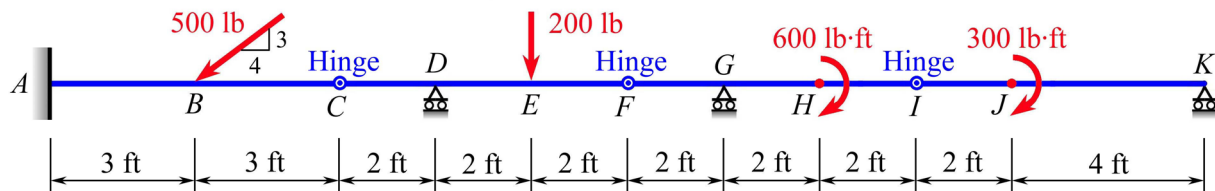
$$\delta U = 0: \quad A_y(\delta y) + 240(-\delta y) + B_y(\delta y) = 0 \quad \therefore B_y = 240 - A_y = 140$$

$$B_y = 140 \text{ lb } \uparrow$$

#### IV. Three Major Steps and One Guiding Strategy in Virtual Work Method: Examples

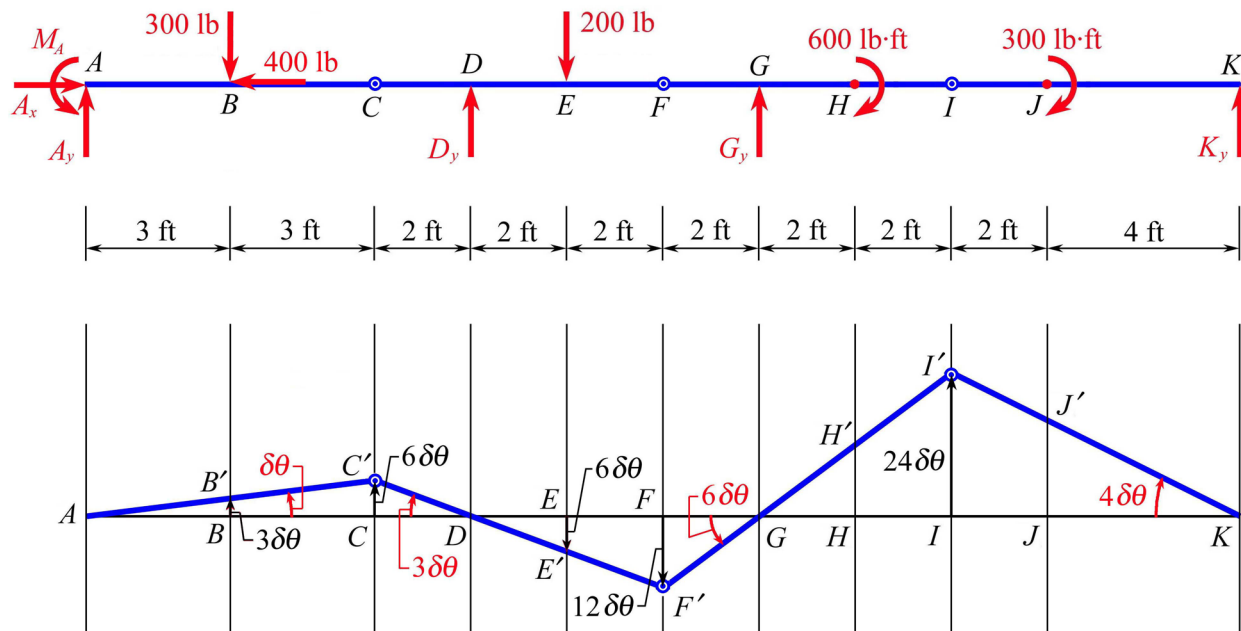
There are **three major steps** in using the *virtual work method*. **Step 1:** Draw the *free-body diagram*. **Step 2:** Draw the *virtual-displacement diagram* with a guiding strategy. **Step 3:** Set to zero the total virtual work done. The **guiding strategy** in step 2 is to give the free body a compatible virtual displacement in such a way that the *one* specified unknown, but *no other unknowns*, will be involved in the virtual work done. That is it: **three major steps** and **one guiding strategy** in the virtual work method! This is demonstrated in the following examples.

**Example 2.** Determine the reaction moment  $M_A$  at the fixed support  $A$  of the combined beam (called a *Gerber beam*) loaded as shown in Fig. 14.



**Fig. 14** A combined beam with hinge connections at  $C$ ,  $F$ , and  $I$

**Solution.** We first draw the **free-body diagram** and a set of compatible virtual displacements for the beam as shown in Fig. 15. Note that we draw this **virtual-displacement diagram** with a **strategy** such that no unknowns *except*  $M_A$  will be involved in the virtual work done.



**Fig. 15** Free-body diagram and virtual-displacement diagram to involve  $M_A$  in  $\delta U = 0$

Referring to Fig. 15 and applying Eq. (8), we directly write

$$\delta U = 0: \quad M_A (\delta\theta) + 300(-3\delta\theta) + 200(6\delta\theta) + 600(-6\delta\theta) + 300(4\delta\theta) = 0$$

$$M_A = 2100$$

$$\mathbf{M}_A = 2100 \text{ lb}\cdot\text{ft} \quad \curvearrowright$$

**Remarks.** With the **conventional method**, we have to refer to the free-body diagram and write

- At hinge  $I$ ,  $M_I = 0$ :  $6K_y - 300 = 0$  (1)

- At hinge  $F$ ,  $M_F = 0$ :  $12K_y - 300 - 600 + 2G_y = 0$  (2)

- At hinge  $C$ ,  $M_C = 0$ :  $18K_y - 300 - 600 + 8G_y - 4(200) + 2D_y = 0$  (3)

- For the entire beam,  $+\circlearrowleft \Sigma M_A = 0$ :

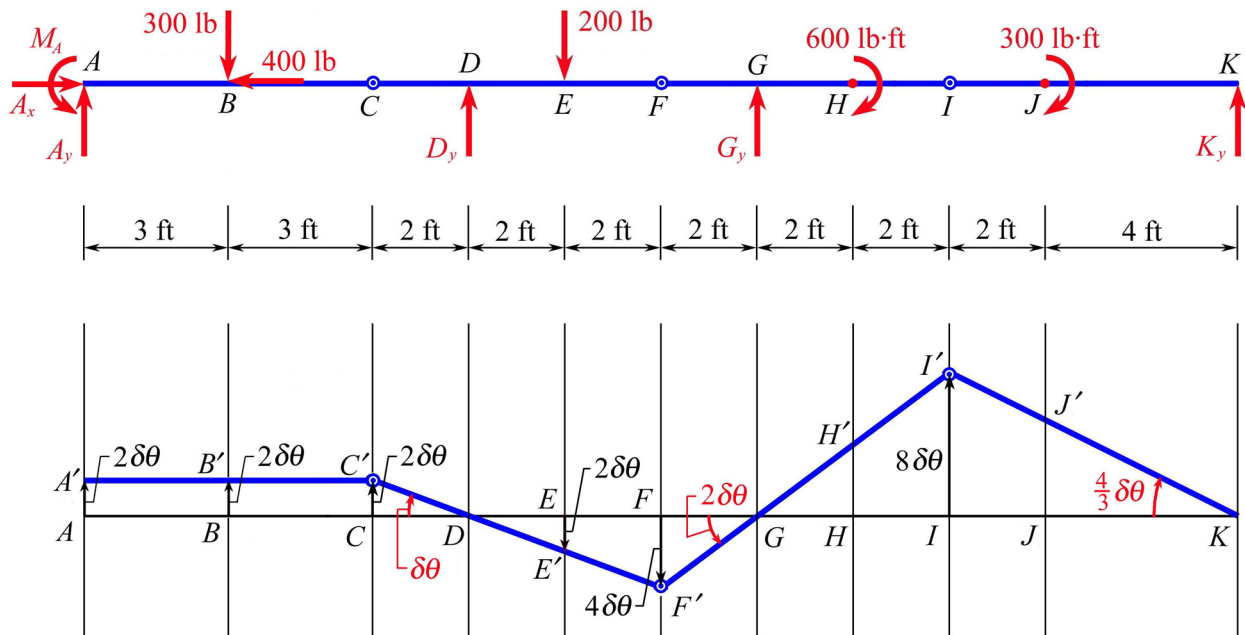
$$M_A - 3(300) + 8D_y - 10(200) + 14G_y - 600 - 300 + 24K_y = 0 \quad (4)$$

These four simultaneous equations yield:  $K_y = 50$ ,  $G_y = 150$ ,  $D_y = -200$ , and  $M_A = 2100$ . Thus, the **conventional method** eventually yields the same solution:

$$\mathbf{M}_A = 2100 \text{ lb}\cdot\text{ft} \quad \curvearrowright$$

**Example 3.** Determine the vertical reaction force  $\mathbf{A}_y$  at the fixed support  $A$  of the combined beam shown in Fig. 14.

**Solution.** We first draw the **free-body diagram** and a set of compatible virtual displacements for the beam as shown in Fig. 16. Note that we draw this **virtual-displacement diagram** with a **strategy** such that no unknowns *except*  $A_y$  will be involved in the virtual work done.



**Fig. 16** Free-body diagram and virtual-displacement diagram to involve  $A_y$  in  $\delta U = 0$

Referring to Fig. 16 and applying Eq. (8), we directly write

$$\delta U = 0: \quad A_y(2\delta\theta) + 300(-2\delta\theta) + 200(2\delta\theta) + 600(-2\delta\theta) + 300\left(\frac{4}{3}\delta\theta\right) = 0$$

$$A_y = 500$$

$$\mathbf{A}_y = 500 \text{ lb} \quad \uparrow$$

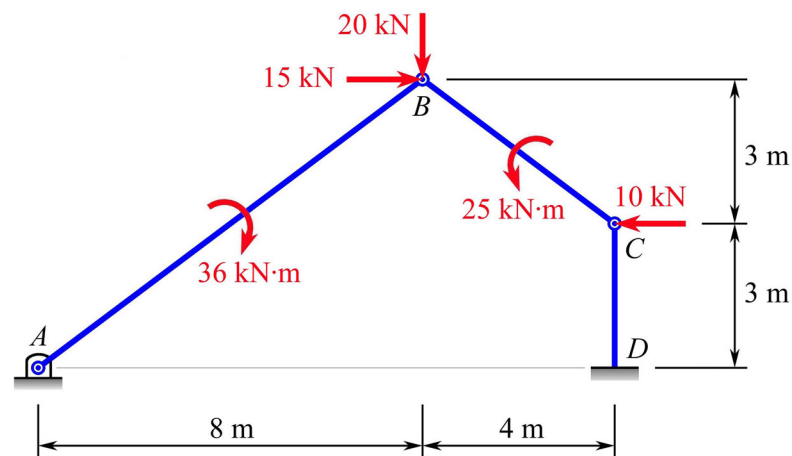


**Remarks.** With the **conventional method**, we have to refer to the free-body diagram and write

- At hinge  $I$ ,  $M_I = 0$ :  $6K_y - 300 = 0$  (1)
- At hinge  $F$ ,  $M_F = 0$ :  $12K_y - 300 - 600 + 2G_y = 0$  (2)
- At hinge  $C$ ,  $M_C = 0$ :  $18K_y - 300 - 600 + 8G_y - 4(200) + 2D_y = 0$  (3)
- For the entire beam,  $+\uparrow \Sigma F_y = 0$ :  $A_y + D_y + G_y + K_y - 300 - 200 = 0$  (4)

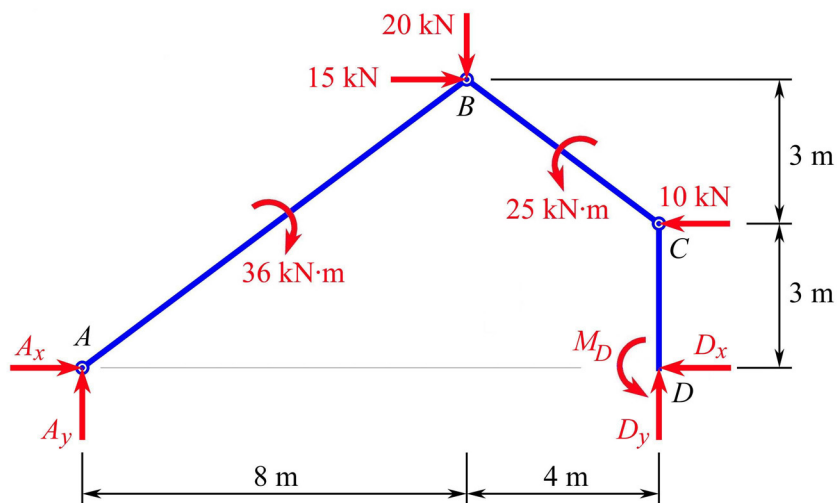
These four simultaneous equations yield:  $K_y = 50$ ,  $G_y = 150$ ,  $D_y = -200$ , and  $A_y = 500$ . Thus, the **conventional method** eventually yields the same solution:  **$A_y = 500 \text{ lb } \uparrow$**

**Example 4.** Determine the horizontal reaction force  $D_x$  at the fixed support  $D$  of the frame loaded as shown in Fig. 17.



**Fig. 17** A frame with hinge support at  $A$  and fixed support at  $D$

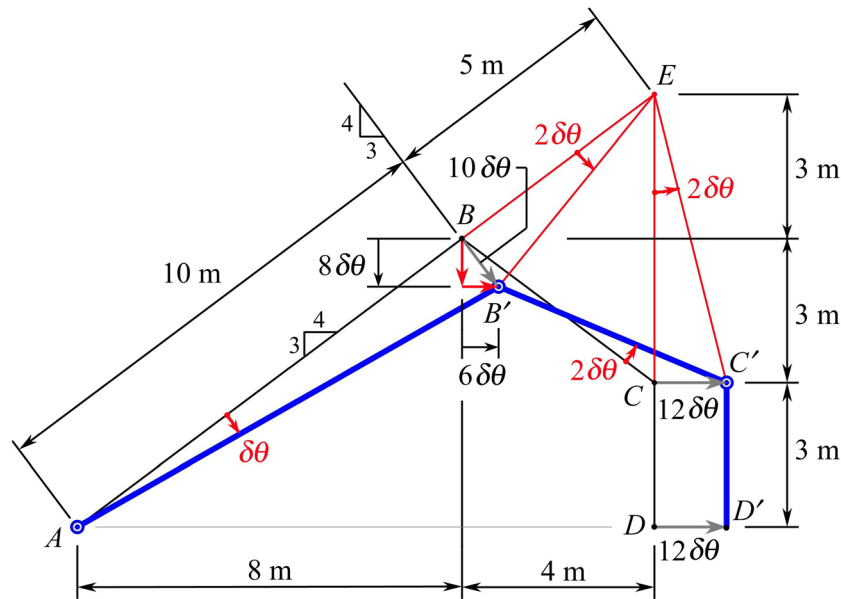
**Solution.** We first draw in Fig. 18 the **free-body diagram** of the frame in Fig. 17.



**Fig. 18** Free-body diagram for the frame

Next, we draw in Fig. 19 a set of compatible virtual displacements for the frame. Note that we draw this **virtual-displacement diagram** with a **strategy** such that no unknowns *except*  $D_x$  will be involved in the virtual work done. In Fig. 19, pay special attention to the following:

- The compatible virtual displacement  $\overline{BB'}$  of point  $B$  is such that  $\overline{BB'} \perp \overline{AB}$  and  $\overline{BB'} = 10\delta\theta$ .
- The displacement center of member  $AB$  is at  $A$ .
- The displacement center of member  $BC$  is at  $E$ .
- The displacement center of member  $CD$  is at  $\infty$ .
- Each of the three sides (i.e.,  $BC$ ,  $CE$ , and  $EB$ ) of the “rigid triangular plate”  $BCE$  rotates counterclockwise through the same angle of  $2\delta\theta$ .
- *Without* benefit of using the displacement center  $E$  of member  $BC$ , the virtual displacement of point  $C$  will need to be determined using **differential calculus** as shown in **Appendix A**.



**Fig. 19** Virtual-displacement diagram to involve  $D_x$  in the virtual work done

Referring to Figs. 18 and 19 and applying Eq. (8), we directly write

$$\delta U = 0: \quad 36(\delta\theta) + 15(6\delta\theta) + 20(8\delta\theta) + 25(2\delta\theta) + 10(-12\delta\theta) + D_x(-12\delta\theta) = 0$$

$$D_x = 18$$

$$\mathbf{D_x = 18 \text{ kN} \leftarrow}$$

**Remarks.** With the **conventional method**, we have to refer to the free-body diagram of the frame in Fig. 18 and write

$$\bullet \text{ At hinge } C, M_C = 0: \quad M_D - 3D_x = 0 \quad (1)$$

$$\bullet \text{ At hinge } B, M_B = 0: \quad M_D - 6D_x + 4D_y - 3(10) + 25 = 0 \quad (2)$$

$$\bullet \text{ For the entire frame, } +\circlearrowleft \sum M_A = 0:$$

$$M_D + 12D_y + 3(10) + 25 - 6(15) - 8(20) - 36 = 0 \quad (3)$$

These three simultaneous equations yield:  $M_D = 54$ ,  $D_x = 18$ , and  $D_y = 14.75$ . Thus, the **conventional method** eventually yields the same solution:

$$\mathbf{D_x = 18 \text{ kN} \leftarrow}$$

## V. Concluding Remarks

Solutions for *simple* equilibrium problems by the virtual work method may come across as “unconventional” when compared with those by the conventional method, as illustrated in Example 1 in Section III. Well, Example 1 was provided merely as a teaching and learning example to bring out head-to-head contrasts between the conventional method and the virtual work method. After all, the virtual work method has been shown as a fabulous method in solving decently challenging problems as illustrated in Examples 2, 3, and 4 in Section IV.

The implementation of the proposed **three major steps** and **one guiding strategy** in the virtual work method, as described and illustrated in Section IV, has greatly helped students understand and implement the virtual work method in one and half weeks at University of Arkansas in the past several years. The enthusiasm of an instructor about the beauty and powerfulness of the virtual work method can readily be contagious to the students. The application of the concept of *displacement center* for each member in a system is what makes possible the use of just algebra and geometry (rather than differential calculus, which is evidenced in **Appendix A**) as prerequisite mathematics for the teaching and learning of the *principle of virtual work* in statics.

Clearly, the advantages of the virtual work method lie in its conciseness in the principle, its visual elegance in the formulation of the solution via virtual-displacement diagrams, and its saving in algebraic effort by doing away with the need to solve simultaneous equations in complex problems. The virtual work method may initially appear as a magic black box to students, but the advantages and elegance witnessed by students are sparks that kindle their interest in learning the virtual work method in particular and the subject of statics in general.

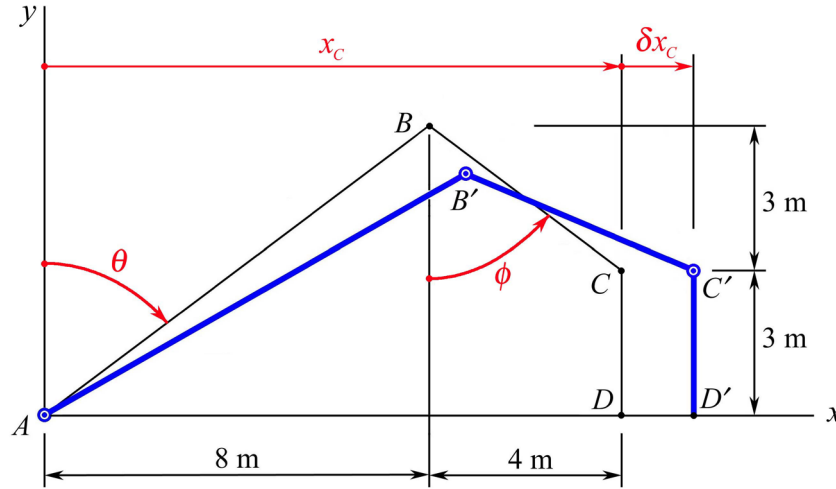
It is true that the drawing of compatible virtual displacements for frames and machines involves basic geometry and requires good graphics skills. These aspects do present some degree of challenges to a number of beginning students. Nevertheless, the learning of the virtual work method is an excellent training ground for engineering and technology students to develop their visual skills in reading technical drawings and presenting technical conceptions.

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## Appendix A: Determination of $\delta x_C$ Using Differential Calculus

Without the benefit of **displacement center**,<sup>5</sup> the determination of the virtual displacement  $\delta x_C$  of the joint C in Fig 19 would not be a matter of just algebra and geometry. We would have to employ **differential calculus** to find it. Recall that the guiding strategy calls for no unknowns *except*  $D_x$  to be involved in the virtual work done. Thus, the center lines of the frame are chosen to undergo a set of virtual displacements (*inconsistent* with constraints) as shown in Fig. 20.



**Fig. 20** Virtual displacement of the frame to involve  $D_x$  in the virtual work done

We first let the angles made by members  $AB$  and  $BC$  with the vertical be  $\theta$  and  $\phi$ , respectively, as indicated in Fig. 20. The constraint on the height of the joint  $C$  is

$$\overline{AB} \cos \theta - \overline{BC} \cos \phi = \overline{CD}$$

Employing **differential calculus**, we write

$$-\overline{AB} (\sin \theta) \delta \theta + \overline{BC} (\sin \phi) \delta \phi = 0 \quad \therefore \delta \phi = \frac{\overline{AB} \sin \theta}{\overline{BC} \sin \phi} \delta \theta = \frac{10(4/5)}{5(4/5)} \delta \theta = 2 \delta \theta$$

The abscissa of the joint  $C$  is

$$x_C = \overline{AB} \sin \theta + \overline{BC} \sin \phi = 10 \sin \theta + 5 \sin \phi$$

Employing **differential calculus**, we write

$$\delta x_C = 10 (\cos \theta) \delta \theta + 5 (\cos \phi) \delta \phi = 10 (3/5) \delta \theta + 5 (3/5) (2 \delta \theta) = 12 \delta \theta$$

$$\delta x_C = 12 \delta \theta$$

Therefore, differential calculus yields the same value for  $\delta x_C$  as that indicated in Fig. 19.

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