AC 2007-610: USING A SINGLE EQUATION TO ACCOUNT FOR ALL LOADS ON A BEAM IN THE METHOD OF DOUBLE INTEGRATION: A CAVEAT

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Using a Single Equation to Account for All Loads on a Beam in the Method of Double Integration: a Caveat

Abstract

When the method of double integration is applied to determine deflections of beams, one has the option of using a single equation containing singularity functions to effectively account for both concentrated and distributed loads on the entire beam without dividing the beam into multiple segments for integrations. This option is a *right way* and an *effective approach* to start the solution for the problem if the beam is a single piece of elastic body with constant flexural rigidity. However, this option becomes a *wrong way* and a *misconception* that will lead to a set of wrong answers if there exists in the beam (e.g., a *combined beam*) a **discontinuity** in *slope* or *flexural rigidity*. Unsuspecting beginners tend to miss the subtlety that a singularity function, like other functions, must have no discontinuity in slope if it is to be integrated or differentiated in its domain. Here, the domain lies along the beam. Since rudiments of singularity functions are a prerequisite background for sensible reading of this paper, they are included as a refresher. The purpose of this paper is to share with educators and practitioners in mechanics a caveat in analyzing hinge-connected beams – a pitfall into which beginners often tumble.

I. Introduction

There are several established methods for determining deflections of beams in mechanics of materials. They include the following:[1-11] *(a)* method of double integration (*with* or *without* the use of singularity functions), *(b)* method of superposition, *(c)* method using moment-area theorems, *(d)* method using Castigliano's theorem, *(e)* conjugate beam method, and *(f)* method using general formulas. Naturally, there are advantages and disadvantages in using any of the above methods. By and large, the method of double integration is a frequently used method in determining slopes and deflections, as well as statically indeterminate reactions at supports, of beams. *Without* use of singularity functions, the method of double integration has an advantage of needing a prerequisite in mathematics only up to simple calculus. However, it has the following drawback: it requires dividing a beam into multiple segments for separate integrations and studies whenever the beam carries concentrated forces or concentrated moments. This means that more constants of integration will be generated in the process of solution, and more boundary conditions will need to be identified and imposed to provide the needed number of equations for the solution.

In this paper, attention is focused on the method of double integration *with* the use of singularity functions. **Mastery** of the definition, integration, and differentiation of **singularity functions**, besides simple calculus, is a *prerequisite* for readers of this paper. For the benefit of a wider readership, a refresher on singularity functions is included in this paper. Readers, who are familiar with the sign conventions in mechanics of materials and the use of singularity functions, may *skip* the *refresher* on the rudiments in the early part (Sects. II and III) of this paper.

II. Sign Conventions for Beams

In the analysis of beams, it is important to adhere to the generally agreed positive and negative signs for loads, shear forces, bending moments, slopes, and deflections of beams. The free-body diagram for a beam *ab* carrying loads is shown in Fig. 1. The positive directions of shear forces $\mathbf{V}_a$ and $\mathbf{V}_b$, moments $\mathbf{M}_a$ and $\mathbf{M}_b$, at ends *a* and *b* of the beam, the concentrated force **P** and concentrated moment **K**, as well as the distributed loads, are illustrated in this figure.

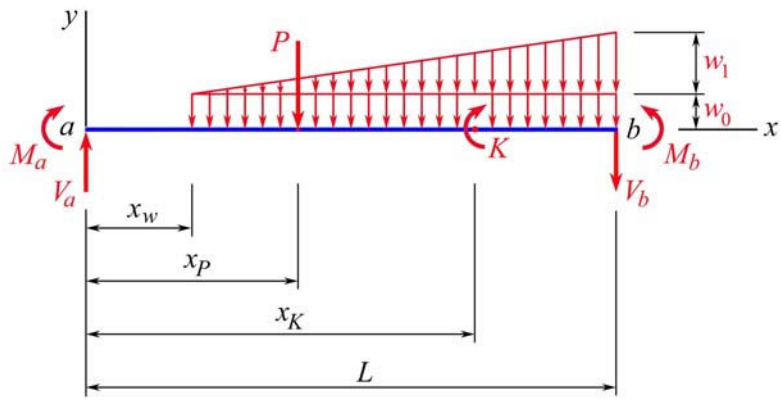

Figure 1. Positive directions of shear forces, moments, and applied loads

In general, we have the following sign conventions for shear forces, moments, and applied loads:

- A *shear force* is *positive* if it acts upward on the left (or downward on the right) face of the beam element (e.g., $\mathbf{V}_a$ at the left end *a*, and $\mathbf{V}_b$ at the right end *b* in Fig. 1).
- At ends of the beam, a *moment* is *positive* if it tends to cause compression in the top fiber of the beam (e.g., $\mathbf{M}_a$ at the left end *a*, and $\mathbf{M}_b$ at the right end *b* in Fig. 1).
- Not at ends of the beam, a *moment* is *positive* if it tends to cause compression in the top fiber of the beam just to the right of the position where it acts (e.g., the concentrated moment **K** in Fig. 1).
- A *concentrated force* or a *distributed force* applied to the beam is *positive* if it is directed downward (e.g., the concentrated force **P**, the uniformly distributed force with intensity w_0, and the linearly varying distributed force with highest intensity w_1 in Fig. 1).

Furthermore, we adopt the following sign conventions for deflection and slope of a beam:

- A *positive deflection* is an upward displacement.
- A *positive slope* is a counterclockwise angular displacement.

III. Singularity functions

As in most textbooks, the argument of a singularity function in this paper is enclosed by angle brackets (i.e., < >), while the argument of a regular function is enclosed by parentheses [i.e., ( )]. The rudiments of singularity functions are summarized as follows:[6-8]

$$< x-a >^n = (x-a)^n \quad \text{if} \quad x-a \geq 0 \quad \text{and} \quad n > 0 \tag{1}$$

$$< x - a >^n = 1 \quad \text{if} \quad x - a \geq 0 \quad \text{and} \quad n = 0 \tag{2}$$

$$< x - a >^n = 0 \quad \text{if} \quad x - a < 0 \tag{3}$$

$$< x - a >^n = 0 \quad \text{if} \quad n < 0 \tag{4}$$

$$\int_{-\infty}^{x} < x - a >^n dx = \frac{1}{n+1} < x - a >^{n+1} \quad \text{if} \quad n > 0 \tag{5}$$

$$\int_{-\infty}^{x} < x - a >^n dx = < x - a >^{n+1} \quad \text{if} \quad n \leq 0 \tag{6}$$

$$\frac{d}{dx} < x - a >^n = n < x - a >^{n-1} \quad \text{if} \quad n > 0 \tag{7}$$

$$\frac{d}{dx} < x - a >^n = < x - a >^{n-1} \quad \text{if} \quad n \leq 0 \tag{8}$$

Equations (2) and (3) imply that, in using singularity functions for beams, we take

$$b^0 = 1 \quad \text{for} \quad b \geq 0 \tag{9}$$

$$b^0 = 0 \quad \text{for} \quad b < 0 \tag{10}$$

Referring to the beam *ab* in Fig. 1, we may, for illustrative purposes, employ the rudiments of singularity functions and observe the defined sign conventions for beams to write the loading function q, the shear force V, and the bending moment M for of this beam as follows:[6-8]

$$\begin{aligned} q = {} & V_a < x >^{-1} + M_a < x >^{-2} - P < x - x_P >^{-1} + K < x - x_K >^{-2} \\ & - w_0 < x - x_w >^0 - \frac{w_1}{L - x_w} < x - x_w >^1 \end{aligned} \tag{11}$$

$$\begin{aligned} V = {} & V_a < x >^0 + M_a < x >^{-1} - P < x - x_P >^0 + K < x - x_K >^{-1} \\ & - w_0 < x - x_w >^1 - \frac{w_1}{2(L - x_w)} < x - x_w >^2 \end{aligned} \tag{12}$$

$$\begin{aligned} M = {} & V_a < x >^1 + M_a < x >^0 - P < x - x_P >^1 + K < x - x_K >^0 \\ & - \frac{w_0}{2} < x - x_w >^2 - \frac{w_1}{6(L - x_w)} < x - x_w >^3 \end{aligned} \tag{13}$$

Any beam element of differential width dx at any position x may be perceived to have a *left face* and a *right face*. Note that Eqs. (11) through (13) are written for the quantities q, V, and M acting on the **left face** of the beam element at any position x, and we have $0 \leq x < L$. Therefore, $x - L < 0$ even though $x \to L$ at the right end of beam. By the definition in Eq. (3), the values of the terms $-V_b < x - L >^{-1} - M_b < x - L >^{-2}$, as well as the integrals of these terms, are always zero for the beam. This is why these terms are trivial and may simply be omitted in the expression for the loading function q in Eq. (11). For further illustration of singularity functions, see Example 1.

Example 1. A cantilever beam AD having a constant flexural rigidity EI carries a concentrated force **P**, a concentrated moment **K**, and a uniformly distributed load of intensity w_0 as shown in Fig. 2, where $\mathbf{P} = w_0 L \downarrow$ and $\mathbf{K} = w_0 L^2 \circlearrowright$. Applying the method of double integration *with* use of singularity functions, determine the slope θ_A and the deflection y_A of the free end A of this beam.

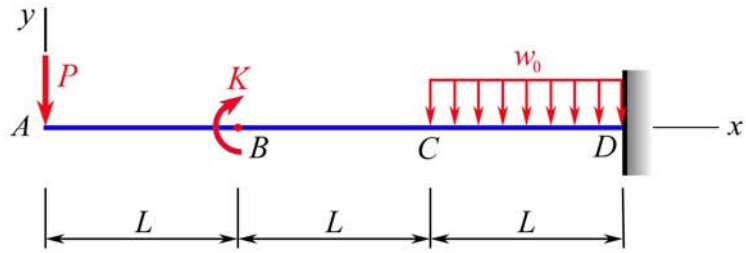

Figure 2. Cantilevered beam with concentrated and distributed loads

Solution: We first write the loading function q, the shear force V, and the bending moment M for the entire beam as follows:

$$q = -P < x >^{-1} + K < x - L >^{-2} - w_0 < x - 2L >^{0}$$

$$V = -P < x >^{0} + K < x - L >^{-1} - w_0 < x - 2L >^{1}$$

$$EIy'' = M = -P < x >^{1} + K < x - L >^{0} - \frac{w_0}{2} < x - 2L >^{2}$$

Double integration of the last equation leads to

$$EIy' = -\frac{P}{2} < x >^{2} + K < x - L >^{1} - \frac{w_0}{6} < x - 2L >^{3} + C_1$$

$$EIy = -\frac{P}{6} < x >^{3} + \frac{K}{2} < x - L >^{2} - \frac{w_0}{24} < x - 2L >^{4} + C_1 x + C_2$$

Imposition of boundary conditions on the beam yields

$$y'(3L) = 0: \qquad 0 = -\frac{P}{2}(3L)^2 + K(2L) - \frac{w_0}{6} L^3 + C_1 \qquad (a)$$

$$y(3L) = 0: \qquad 0 = -\frac{P}{6}(3L)^3 + \frac{K}{2}(2L)^2 - \frac{w_0}{24} L^4 + C_1(3L) + C_2 \qquad (b)$$

Using the given value of P and K and solving the above *two* simultaneous equations, we write

$$P = w_0 L \qquad\qquad K = w_0 L^2$$

$$C_1 = \frac{9PL^2}{2} - 2KL + \frac{w_0 L^3}{6} = \frac{8 w_0 L^3}{3} \qquad\qquad C_2 = -9PL^3 + 4KL^2 - \frac{11 w_0 L^4}{24} = -\frac{131 w_0 L^4}{24}$$

Substituting the above solutions into foregoing equations for EIy' and EIy, we write

$$\theta_A = y'(0) = \frac{C_1}{EI} = \frac{8 w_0 L^3}{3EI} \qquad\qquad \theta_A = \frac{8 w_0 L^3}{3EI} \circlearrowleft$$

$$y_A = y(0) = \frac{C_2}{EI} = -\frac{131 w_0 L^4}{24EI} \qquad\qquad y_A = \frac{131 w_0 L^4}{24EI} \downarrow$$

IV. Analysis of a Hinge-Connected Beam: Wrong and Right Ways

Employing singularity functions, one can often use a single equation to account for all loads acting on the entire beam [e.g., Eqs. (11), (12), and (13) for the loads shown in Fig. 1]. However, most textbooks for mechanics of materials or mechanical design do not provide explicit warning that one **cannot** use a single equation containing singularity functions to blaze the various loads on the **entire** beam when the beam under loading has a *discontinuity* in its *slope*. In fact, even singularity functions cannot be exempt from the rule that a well-behaved function must have continuous slope in its domain if it is to be integrated or differentiated in that domain. For a beam, the domain lies along the beam. If a beam is composed of two or more segments that are connected by hinges (as in a *Gerber beam*), then the beam has discontinuity in slope at the hinge connections when loads are applied to act on the beam. In such a case, the deflections and any statically indeterminate reactions must be analyzed by dividing the beam into segments, each of which must have *no* discontinuity in slope. Otherwise, erroneous results will be reached.

Example 2. A beam AE with a hinge connector at C carries a concentrated force $\mathbf{P}$ at D and is supported as shown in Fig. 3, where the segments AC and CE have the same flexural rigidity EI. An unsuspecting beginner, who tries to apply the method of double-integration *with* the use of singularity functions, arrived at a set of wrong answers for *(a)* the reaction moment $\mathbf{M}_A$ and the vertical reaction force $\mathbf{A}_y$ at A, and *(b)* the vertical reaction force $\mathbf{B}_y$ at B. What may be the likely **wrong way** taken by this person?

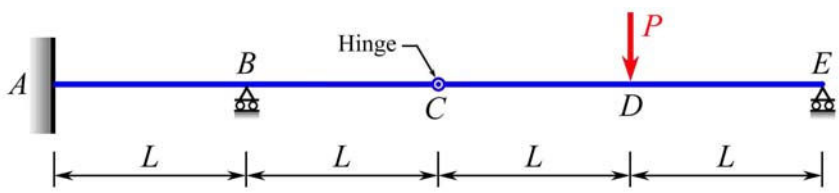

Figure 3. Hinge-connected beam with a fixed end and two simple supports

Solution – wrong way: Let us assume that this person has drawn a correct free-body diagram of the beam, as shown in Fig. 4, in the beginning of the solution.

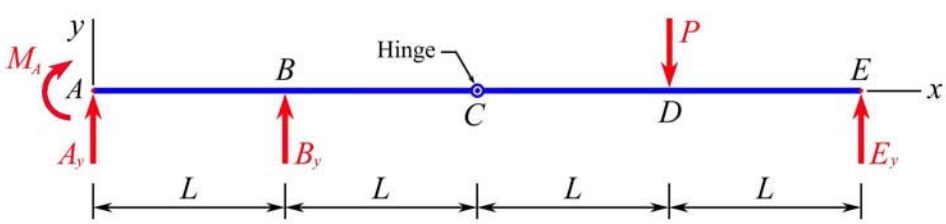

Figure 4. Free-body diagram with assumption of positive reaction forces and moments

This beam is statically indeterminate to the *first* degree. Due to lack of adequate warning on the case of a beam with discontinuity in slope, this person is likely of the impression or opinion that, by employing singularity functions, one can "always" use a single equation to account for all loads acting on the entire beam. Therefore, this person uses singularity functions to blaze the loading on the free-body diagram in Fig. 4 to first write the loading function q, the shear force V, and the bending moment M for the entire beam as follows:

$$q = M_A < x >^{-2} + A_y < x >^{-1} + B_y < x - L >^{-1} - P < x - 3L >^{-1}$$

$$V = M_A < x >^{-1} + A_y < x >^{0} + B_y < x - L >^{0} - P < x - 3L >^{0}$$

$$EI\,y'' = M = M_A < x >^{0} + A_y < x >^{1} + B_y < x - L >^{1} - P < x - 3L >^{1}$$

Double integration of the last equation leads to

$$EIy' = M_A < x >^{1} + \frac{1}{2} A_y < x >^{2} + \frac{1}{2} B_y < x - L >^{2} - \frac{1}{2} P < x - 3L >^{2} + C_1$$

$$EIy = \frac{1}{2} M_A < x >^{2} + \frac{1}{6} A_y < x >^{3} + \frac{1}{6} B_y < x - L >^{3} - \frac{1}{6} P < x - 3L >^{3} + C_1 x + C_2$$

Imposition of boundary conditions on the beam yields

$y'(0) = 0$: $\quad 0 = C_1$ *(a)*

$y(0) = 0$: $\quad 0 = C_2$ *(b)*

$y(L) = 0$: $\quad 0 = \frac{1}{2} M_A L^2 + \frac{1}{6} A_y L^3 + C_1 L + C_2$ *(c)*

$y(4L) = 0$: $\quad 0 = \frac{1}{2} M_A (4L)^2 + \frac{1}{6} A_y (4L)^3 + \frac{1}{6} B_y (3L)^3 - \frac{1}{6} PL^3 + C_1 (4L) + C_2$ *(d)*

Equilibrium of the entire beam in Fig. 4 gives

$+\circlearrowleft \Sigma M_E = 0$: $\quad -M_A - 4LA_y - 3LB_y + LP = 0$ *(e)*

Solution of the above *five* simultaneous equations in *(a)* through *(e)* yields

$$C_1 = 0 \qquad C_2 = 0 \qquad M_A = \frac{8PL}{45} \qquad A_y = -\frac{8P}{15} \qquad B_y = \frac{133P}{135}$$

Consistent with the defined sign conventions, this unsuspecting beginner is led to report

$$\mathbf{M}_A = \frac{8PL}{45} \circlearrowright \qquad \mathbf{A}_y = \frac{8P}{15} \downarrow \qquad \mathbf{B}_y = \frac{133P}{135} \uparrow$$

According to these seeming "answers," which satisfy Eq. *(e)*, the moment at *C* in Fig. 4 would be

$$M_C = M_A + 2LA_y + LB_y = \frac{8PL}{45} + 2L\left(-\frac{8P}{15}\right) + L\left(\frac{133P}{135}\right) = \frac{13PL}{135} \neq 0$$

Since the **moment at a hinge must be zero** (i.e., $M_C = 0$), the above answers must be wrong!

Example 3. A beam *AE* with a hinge connector at *C* carries a concentrated force **P** at *D* and is supported as shown in Fig. 3, where the segments *AC* and *CE* have the same flexural rigidity *EI*. Show the **right way** to apply the method of double integration *with* the use of singularity functions to determine for this beam *(a)* the reaction moment $\mathbf{M}_A$ and the vertical reaction force $\mathbf{A}_y$ at *A*, *(b)* the vertical reaction force $\mathbf{B}_y$ at *B*, *(c)* the deflection y_C of the hinge at *C*, *(d)* the slopes θ_{CL} and θ_{CR} just to the left and just to the right of the hinge at *C*, respectively, and *(e)* the slope θ_D and the deflection y_D at *D*.

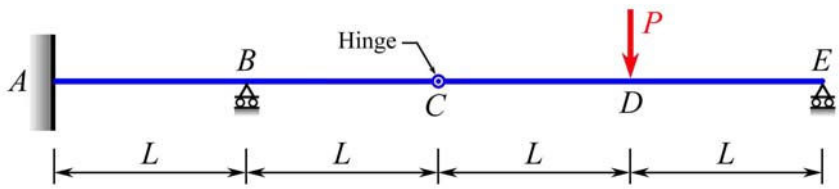

Figure 3. Hinge-connected beam with a fixed end and two simple supports (repeated)

Solution – right way: This beam is statically indeterminate to the *first* degree. Because of the discontinuity in slope at the hinge connection *C*, this beam needs to be divided into two segments *AC* and *CE* for analysis in the solution, where no discontinuity in slope exists in either segment.

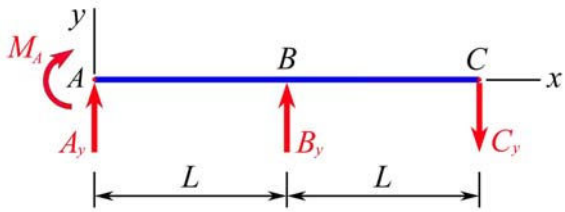

Figure 5. Free-body diagram for segment *AC*

The loading function q_{AC}, the shear force V_{AC}, and the bending moment M_{AC} for the segment *AC*, as shown in Fig. 5, are

$$q_{AC} = M_A < x >^{-2} + A_y < x >^{-1} + B_y < x - L >^{-1}$$

$$V_{AC} = M_A < x >^{-1} + A_y < x >^{0} + B_y < x - L >^{0}$$

$$EIy''_{AC} = M_{AC} = M_A < x >^{0} + A_y < x >^{1} + B_y < x - L >^{1}$$

Double integration of the last equation leads to

$$EIy'_{AC} = M_A < x >^{1} + \frac{1}{2} A_y < x >^{2} + \frac{1}{2} B_y < x - L >^{2} + C_1$$

$$EIy_{AC} = \frac{1}{2} M_A < x >^{2} + \frac{1}{6} A_y < x >^{3} + \frac{1}{6} B_y < x - L >^{3} + C_1 x + C_2$$

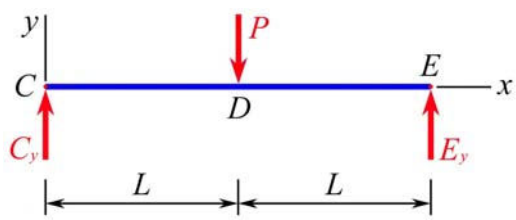

Figure 6. Free-body diagram for segment *CE*

The loading function q_{CE}, the shear force V_{CE}, and the bending moment M_{CE} for the segment *CE*, as shown in Fig. 6, are

$$q_{CE} = C_y < x >^{-1} - P < x - L >^{-1}$$

$$V_{CE} = C_y < x >^{0} - P < x - L >^{0}$$

$$EI\,y''_{CE} = M_{CE} = C_y < x >^1 - P < x - L >^1$$

Double integration of the last equation leads to

$$EI\,y'_{CE} = \frac{1}{2} C_y < x >^2 - \frac{1}{2} P < x - L >^2 + C_3$$

$$EI y_{CE} = \frac{1}{6} C_y < x >^3 - \frac{1}{6} P < x - L >^3 + C_3\,x + C_4$$

Imposition of boundary conditions on the beam segments AC and CE yields

$y'_{AC}(0) = 0$: $\quad 0 = C_1$ *(a)*

$y_{AC}(0) = 0$: $\quad 0 = C_2$ *(b)*

$y_{AC}(L) = 0$: $\quad 0 = \frac{1}{2} M_A (L)^2 + \frac{1}{6} A_y (L)^3 + C_1 L + C_2$ *(c)*

$y_{AC}(2L) = y_{CE}(0)$: $\quad \frac{1}{2} M_A (2L)^2 + \frac{1}{6} A_y (2L)^3 + \frac{1}{6} B_y (L)^3 + C_1 (2L) + C_2 = C_4$ *(d)*

$y_{CE}(2L) = 0$: $\quad 0 = \frac{1}{6} C_y (2L)^3 - \frac{1}{6} P (L)^3 + C_3 (2L) + C_4$ *(e)*

Equilibrium for segment AC in Fig. 5 gives

$+\circlearrowleft \Sigma M_C = 0$: $\quad -M_A - 2L A_y - L B_y = 0$ *(f)*

$+\uparrow \Sigma F_y = 0$: $\quad A_y + B_y - C_y = 0$ *(g)*

Equilibrium for segment CE in Fig. 6 gives

$+\circlearrowleft \Sigma M_E = 0$: $\quad -2L C_y + L P = 0$ *(h)*

Solution of the above *eight* simultaneous equations in *(a)* through *(h)* yields

$$C_1 = 0 \qquad C_2 = 0 \qquad C_3 = -\frac{5PL^2}{48} \qquad C_4 = -\frac{7PL^3}{24}$$

$$M_A = \frac{PL}{4} \qquad A_y = -\frac{3P}{4} \qquad B_y = \frac{5P}{4} \qquad C_y = \frac{P}{2}$$

Consistent with the defined sign conventions, we report that

$$\mathbf{M}_A = \frac{PL}{4} \circlearrowright \qquad \mathbf{A}_y = \frac{3P}{4} \downarrow \qquad \mathbf{B}_y = \frac{5P}{4} \uparrow$$

(These answers are obtained in a *right way* and are **different** from those obtained earlier for $\mathbf{M}_A$, $\mathbf{A}_y$, and $\mathbf{B}_y$ in a *wrong way* by an unsuspecting beginner in Example 1.)

Substituting the above obtained values into the equations for EIy_{CE}, EIy'_{AC}, and EIy'_{CE}, we write

$$EIy_C = EIy_{CE}(0) = C_4 = -\frac{7PL^3}{24} \qquad y_C = \frac{7PL^3}{24EI} \downarrow$$

$$EI\theta_{CL} = EIy'_{AC}(2L) = M_A(2L) + \frac{1}{2}A_y(2L)^2 + \frac{1}{2}B_y(L)^2 + C_1 = -\frac{3PL^2}{8} \qquad \theta_{CL} = \frac{3PL^2}{8EI} \circlearrowright$$

$$EI\theta_{CR} = EIy'_{CE}(0) = C_3 = -\frac{5PL^2}{48} \qquad \theta_{CR} = \frac{5PL^2}{48EI} \circlearrowright$$

$$EI\theta_D = EIy'_{CE}(L) = \frac{1}{2}C_yL^2 + C_3 = \frac{7PL^2}{48} \qquad \theta_D = \frac{7PL^2}{48EI} \circlearrowleft$$

$$EIy_D = EIy_{CE}(L) = \frac{1}{6}C_yL^3 + C_3L + C_4 = -\frac{5PL^3}{16} \qquad y_D = \frac{5PL^3}{16EI} \downarrow$$

Based on the preceding solutions, the slopes and deflections of the hinge-connected beam *AE* may be plotted as illustrated in Fig. 7, where one can readily appreciate the different slopes θ_{CL} and θ_{CR} at *C*.

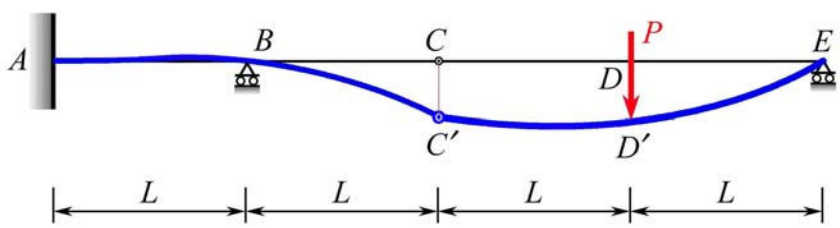

Figure 7. Deflections of the beam *AE*

The foregoing results and answers are obtained by the method of double integration *with* the use of singularity functions via a **right way**. These answers have been *assessed* and *verified* to be in agreement with the answers that were independently obtained for a problem involving the same beam but being solved using an entirely different method – the conjugate beam method.[10]

V. Conclusion

There are advantages and disadvantages in using any of the several established methods for analyzing deflections of beams. The aim of this paper is to share with educators and practitioners in mechanics a **caveat** to avoid a common unsuspected pitfall when applying the method of double integration *with* the use of singularity functions to solve problems involving slopes and deflections, as well as statically indeterminate reactions at supports, of beams. The paper is *not* written to advocate this particular method over other established methods.

For the benefit of a wider readership, the paper goes over the sign conventions for beams and the rudiments of singularity functions as applied to the analysis of beams. Most textbooks for mechanics of materials or mechanical design do not adequately warn readers about the *limitations*

of singularity functions and the *pitfall* in the case of hinge-connected beams, where discontinuity in slope of the beam exists. Beginning students tend to be of the impression that singularity functions are a powerful mathematical tool that will enable them to use a single equation to account for both concentrated and distributed loads on the entire beam without the need to divide it into segments for analysis *in all cases*. Such an impression is a correct one if the beam is a single piece of elastic body that has a constant flexural rigidity, but it is a misconception for the analysis of a hinge-connected beam. Thus, a hinge-connected beam is a pitfall into which unsuspecting persons often tumble.

It is emphasized in the paper that singularity functions cannot be exempt from the mathematical rule that requires a function to have continuous slope in a domain if it is to be integrated or differentiated in that domain. Here, the domain lies along the beam. The paper includes illustrative examples to demonstrate both **wrong** and **right** ways in using singularity functions in the method of double integration to solve a problem involving a hinge-connected beam. In general, deflections and any statically indeterminate reactions associated with a hinge-connected beam must be analyzed by dividing the beam into segments, as required, where each segment must have no discontinuity in slope. Otherwise, erroneous results will be reached.

VI. References

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