AC 2008-2796: DEFLECTION OF A BEAM IN NEUTRAL EQUILIBRIUM À LA CONJUGATE BEAM METHOD: USE OF SUPPORT, NOT BOUNDARY, CONDITIONS

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Deflection of a Beam in Neutral Equilibrium à la Conjugate Beam Method: Use of Support, *Not* Boundary, Conditions

Abstract

Beams with flexural rigidity will deflect under loading. Is it possible to ascertain the deflection of a loaded beam in neutral equilibrium? The answer is *yes* according to the *conjugate beam method*, but a resounding *no* according to all other established methods. The objective of this paper is to share with fellow engineering educators the insights, highlights, and several illustrative examples for teaching the conjugate beam method. In particular, it is pointed out that *(a)* support conditions (*or* types), *rather than* boundary conditions, are what the conjugate beam method needs in finding solutions for deflections of loaded beams, *(b)* more *support* conditions than *boundary* conditions are usually known for beams in neutral equilibrium, and *(c)* the conjugate beam method often *works better than* other established methods in determining deflections of beams. It is demonstrated in this paper that the conjugate beam method does find the likely, *or* unique, deflection of a loaded beam in neutral equilibrium.

I. Introduction

All beams considered in this paper are elastic beams, which are longitudinal members subjected to transverse loads and are usually in static equilibrium. A beam is in *neutral* equilibrium if the force system acting on the beam is statically balanced and the potential energy of the beam in the neighborhood of its equilibrium configuration is constant.

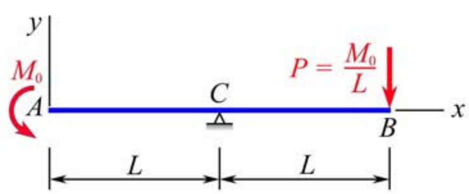

Fig. 1 Actual beam

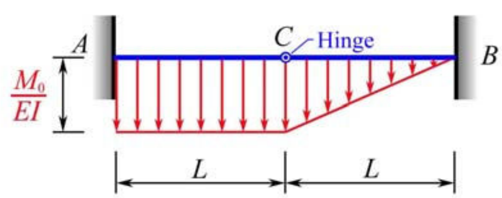

Fig. 2 Conjugate beam

The beam in Fig. 1 is in neutral equilibrium and will adopt a deflected shape. Is it possible to ascertain the deflection of a loaded beam in neutral equilibrium? The answer is *yes* according to the conjugate beam method,[1–4] but a resounding *no* according to all other established methods,[3–12] such as *(a)* method of double integration (*with* or *without* the use of singularity functions), *(b)* method of superposition, *(c)* method using moment-area theorems, *(d)* method using Castigliano's theorem, and *(e)* method of segments. These other methods all expect a beam to have *sufficient* well-defined boundary conditions for use in seeking a unique solution for the deflection of the beam. The beam in Fig. 1 manifests only *one* known boundary condition (i.e., the deflection at the hinge support C is zero), which is simply *insufficient* to allow the other methods to settle on a unique solution. However, the conjugate beam method has no trouble with the beam in Fig. 1. This beam manifests *three* support conditions (i.e., *free end* at A, *simple support* at C, and *free end* at B), which are sufficient to allow a corresponding conjugate beam to be constructed as shown in Fig. 2. For now, the actual beam and the conjugate beam in Figs. 1 and 2 are used to serve introductory purposes. Finding of the deflection of this beam in neutral equilibrium is deferred to Example 5 later.

The conjugate beam method was first propounded in 1921 by Westergaard.[1] One can find brief presentations of this method in earlier mechanics of materials textbooks by Timoshenko and MacCullough[3] and by Singer and Pytel[4]. Recently, a set of ten guiding rules to facilitate the use of this method was synthesized by Jong[2] from the original paper of Westergaard.[1] For benefit of a wider readership with different specialties, a brief summary of the guiding rules needed in this method is included. Readers, who are familiar with the rudiments of this method, may *skip* the part presented in Section II.

This paper is intended to share with fellow engineering educators the insights, highlights, and several illustrative examples for teaching students that *(a)* the conjugate beam method is a natural and logical extension of the method using moment-area theorems, *(b)* by observing the support conditions of an actual beam (as in Fig. 1) and applying the guiding rules in this method, the conjugate beam for an actual beam can readily be constructed (as in Fig. 2), *(c)* slopes and deflections of an actual beam are simply obtained from the "shearing forces" and "bending moments" in the corresponding positions of the conjugate beam, *(d)* support conditions, rather than boundary conditions, are what the conjugate beam method needs in generating solutions for slopes and deflections of actual beams, *(e)* more support conditions than boundary conditions are usually known for beams in neutral equilibrium, and *(f)* the conjugate beam method can do whatever other established methods can do, and more, in determining deflections of beams. The paper demonstrates that the conjugate beam method does find the unique deflection of a loaded beam in neutral equilibrium. The results obtained are assessed analytically by comparison with well-known results in textbooks.

II. Guiding Rules in Conjugate Beam Method

Although Westergaard[1] propounded the conjugate beam method in a 28-page paper, earlier textbooks[3,4] provided mainly brief and elementary presentations of this method. Without adequate guiding rules and a good number of typical examples, the conjugate beam method may likely appear as inaccessible or esoteric to many beginners. On the other hand, most beginners are pleasantly surprised to learn that there are only two major steps in this method. The *first step* is to set up an additional beam, called *conjugate beam*, besides the actual beam. The *second step* is to determine the "**shearing forces**" and "**bending moments**" in the conjugate beam using mainly concepts and skills in statics. In the process, these two steps are most effectively guided by the set of ten rules synthesized by Jong.[2] These rules are natural and logical extensions of the method using moment-area theorems.[11] They are summarized as follows:

> **Rule 1:** The conjugate beam and the actual beam are of the **same length**.
>
> **Rule 2:** The **loading** on the conjugate beam is simply the distributed **elastic weight**, which is given by the bending moment M in the actual beam divided by the flexural rigidity EI of the actual beam. (The *elastic weight*, M/EI, points upward if the *bending moment* is positive — to cause the top fiber in compression — in beam convention.)

For each *existing support condition* of the actual beam, there is a *corresponding support condition* for the conjugate beam. The correspondence is given by **rules 3** through **7** listed in Table 1, where a *simple support* is either a roller support or a hinge support, since a beam is usually not subjected to axial loads.

Table 1 Corresponding support condition for the conjugate beam

	Existing support condition of the **actual beam**	Corresponding support condition for the **conjugate beam**
Rule 3:	Fixed end	Free end
Rule 4:	Free end	Fixed end
Rule 5:	Simple support at the end	Simple support at the end
Rule 6:	Simple support *not* at the end	*Unsupported* hinge
Rule 7:	*Unsupported* hinge	Simple support

The *slope* and *deflection* of the actual beam are obtained by employing the following rules:

Rule 8: The conjugate beam (hence its free body) is in static **equilibrium**.

Rule 9: The **slope** of (the centerline of) the actual beam at any cross section is given by the "**shearing force**" at that cross section of the conjugate beam. (This **slope** is positive, or counterclockwise, if the "**shearing force**" is positive — tending to rotate the beam element clockwise — in beam convention.)

Rule 10: The **deflection** of (the centerline of) the actual beam at any point is given by the "**bending moment**" at that point of the conjugate beam. (This **deflection** is upward if the "**bending moment**" is positive — tending to cause the top fiber in compression — in beam convention.)

III. Applications of Conjugate Beam Method

For better understanding of the method, this section includes several detailed examples with different degrees of complexity and challenge. In applications of the conjugate beam method, more statics skills and very little explicit calculus skills are usually needed. With the guiding rules, as summarized in Section II, one may here see that the conjugate beam method is accessible and easy to apply to find deflections of beams.

Example 1. A cantilever beam AB with total length $2L$ and constant flexural rigidity EI carries a concentrated force $P\downarrow$ at C as shown in Fig. 3. Determine *(a)* the slope θ_A and deflection y_A at A, *(b)* the slope θ_C and deflection y_C at C.

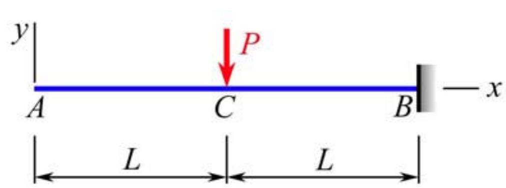

Fig. 3 A cantilever beam (actual beam)

Solution. In accordance with **rules 1** through **4** in Section II, we first construct in Fig. 4 the conjugate beam (i.e., the *additional beam*) for the actual beam in Fig. 3. Notice that the conjugate beam has the same length $2L$ as the actual beam, and it carries a linearly distributed *elastic weight* pointing downward with intensity equal to zero at C and equal to PL/EI at B.

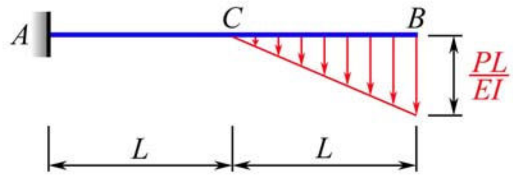

Fig. 4 Conjugate beam (additional beam) for the beam in Fig. 3

Furthermore, notice that the actual beam in Fig. 3 has a free end at A and a fixed end at B, while the conjugate beam in Fig. 4 has a fixed end at A and a free end at B. Next, we draw in Fig. 5 the free-body diagram for the conjugate beam, where the superscript ^{c} is used to signify a quantity associated with the *conjugate beam*. Such a notation is needed for distinguishing a quantity associated with the conjugate beam from the force or moment acting on the actual beam.

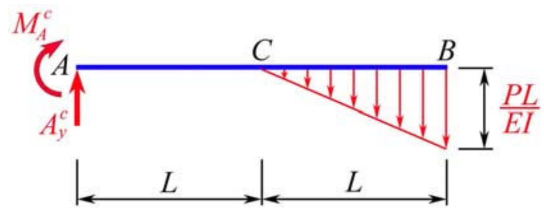

Fig. 5 Free-body diagram for the conjugate beam in Fig. 4

By **rule 8** in Section II, the conjugate beam (hence its free body) is in static **equilibrium**. Referring to Fig. 5, we write

$$+\uparrow \Sigma F_y^c = 0: \qquad A_y^c - \frac{L}{2}\cdot\frac{PL}{EI} = 0 \qquad A_y^c = \frac{PL^2}{2EI} \qquad \mathbf{A}_y^c = \frac{PL^2}{2EI}\uparrow$$

$$+\circlearrowleft \Sigma M_A^c = 0: \qquad -M_A^c - \left(L + \frac{2L}{3}\right)\cdot\frac{PL^2}{2EI} = 0 \qquad M_A^c = -\frac{5PL^3}{6EI} \qquad \mathbf{M}_A^c = \frac{5PL^3}{6EI}\circlearrowleft$$

By **rules 9** and **10** in Section II, the slope θ_A and the deflection y_A at the free end A of the actual beam in Fig. 3 are, respectively, given by the "shearing force" $\mathbf{V}_A^c$ and the "bending moment" $\mathbf{M}_A^c$ at the fixed end A of the conjugate beam in Fig. 4. We write

$$\theta_A = V_A^c = A_y^c = \frac{PL^2}{2EI} \qquad\qquad y_A = M_A^c = -\frac{5PL^3}{6EI}$$

We report that

$$\theta_A = \frac{PL^2}{2EI}\circlearrowleft \qquad\qquad y_A = \frac{5PL^3}{6EI}\downarrow$$

Note that the slope θ_A is *counterclockwise* because the "shearing force" $\mathbf{A}_y^c$ tends to rotate the beam element at A clockwise and is positive, while the deflection y_A is *downward* because the "bending moment" $\mathbf{M}_A^c$ tends to cause *tension* in the top fiber of the beam at A and is negative. Applying **rules 9** and **10** in Section II and referring to Fig. 5, we determine the slope θ_C and deflection y_C at C as follows:

$$\theta_C = V_C^c = A_y^c = \frac{PL^2}{2EI} \qquad\qquad y_C = M_C^c = M_A^c + LA_y^c = -\frac{5PL^3}{6EI} + \frac{PL^3}{2EI} = -\frac{PL^3}{3EI}$$

We report that

$$\theta_C = \frac{PL^2}{2EI} \circlearrowleft \qquad\qquad y_C = \frac{PL^3}{3EI} \downarrow$$

These results are illustrated in Fig. 6.

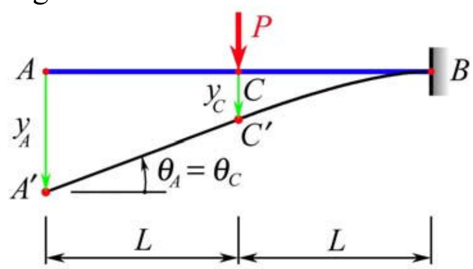

Fig. 6 Deflection of the cantilever beam in Fig. 3

Remark: The preceding results for the cantilever beam *AB* in Fig. 3 are *in agreement* with slopes and deflections for cantilever beams contained in a table *or* an appendix of textbooks.[4,11]

Example 2. A simply supported beam *AB* with length *L* and constant flexural rigidity *EI* carries a concentrated force $P\downarrow$ at *C* as shown in Fig. 7. Determine *(a)* the slopes θ_A, θ_B, θ_C at *A*, *B*, *C*; *(b)* the deflection y_C at *C*.

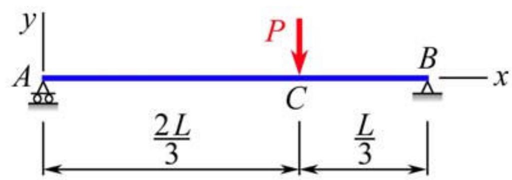

Fig. 7 A simply supported beam (actual beam)

Solution. The reactions at the simple supports *A* and *B* of this beam can readily be determined from statics equilibrium of the beam *AB* and are shown in the free-body diagram in Fig. 8.

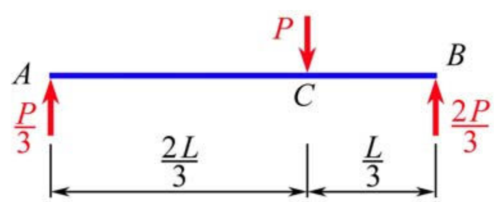

Fig. 8 Free-body diagram for the beam in Fig. 7

Applying **rules 1, 2,** and **5** in Section II, we readily construct in Fig. 9 the conjugate beam for the actual beam in Fig. 7, where the *simple supports* at the ends remain *unchanged* and the loading diagram showing the distributed *elastic weight* on the conjugate beam is drawn *by parts*.

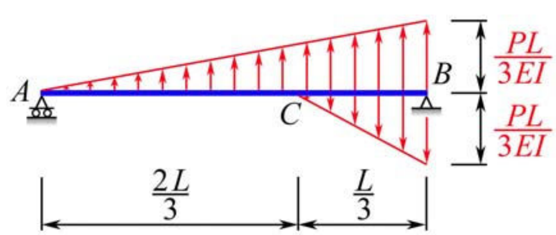

Fig. 9 Conjugate beam (additional beam) for the beam in Fig. 7

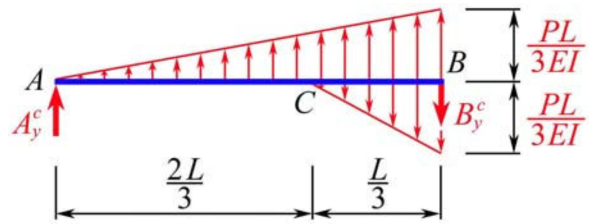

Fig. 10 Free-body diagram for the conjugate beam in Fig. 9

By **rule 8** in Section II, the conjugate beam (hence its free body) is in static **equilibrium**. Referring to Fig. 10, we write

$$+\circlearrowleft \Sigma M_B^c = 0: \quad -LA_y^c - \frac{L}{3}\left(\frac{1}{2}\cdot L\cdot\frac{PL}{3EI}\right) + \frac{L}{9}\left(\frac{1}{2}\cdot\frac{L}{3}\cdot\frac{PL}{3EI}\right) = 0 \qquad A_y^c = -\frac{4PL^2}{81EI}$$

$$+\uparrow \Sigma F_y^c = 0: \quad A_y^c + \frac{1}{2}\cdot L\cdot\frac{PL}{3EI} - \frac{1}{2}\cdot\frac{L}{3}\cdot\frac{PL}{3EI} - B_y^c = 0 \qquad B_y^c = \frac{5PL^2}{81EI}$$

By **rule 9** in Section II, slopes θ_A and θ_B of the actual beam at A and B are given by "shearing forces" $\mathbf{V}_A^c$ and $\mathbf{V}_B^c$ in the conjugate beam at A and B, respectively. We write

$$\theta_A = V_A^c = A_y^c = -\frac{4PL^2}{81EI} \qquad \theta_B = V_B^c = B_y^c = \frac{5PL^2}{81EI}$$

Applying **rules 9** and **10** in Section II and referring to Fig. 10, we determine the slope θ_C and deflection y_C at C as follows:

$$\theta_C = V_C^c = A_y^c + \frac{1}{2}\cdot\frac{2L}{3}\cdot\left(\frac{2}{3}\cdot\frac{PL}{3EI}\right) = \frac{2PL^2}{81EI}$$

$$y_C = M_C^c = \frac{2L}{3}\cdot A_y^c + \frac{2L}{9}\cdot\left(\frac{1}{2}\cdot\frac{2L}{3}\cdot\frac{2}{3}\cdot\frac{PL}{3EI}\right) = -\frac{4PL^3}{243EI}$$

We report that

$$\boxed{\theta_A = \frac{4PL^2}{81EI} \circlearrowright} \qquad \boxed{\theta_B = \frac{5PL^2}{81EI} \circlearrowleft} \qquad \boxed{\theta_C = \frac{2PL^2}{81EI} \circlearrowleft} \qquad \boxed{y_C = \frac{4PL^3}{243EI} \downarrow}$$

These results are illustrated in Fig. 11.

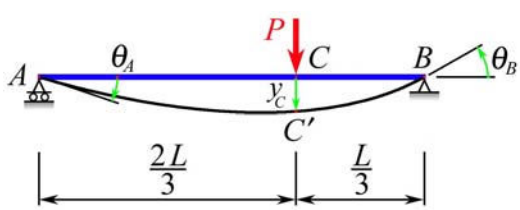

Fig. 11 Deflection of the simply supported beam in Fig. 7

Remark: The preceding results are *in agreement* with slopes and deflections for simple beams contained in a table *or* an appendix of textbooks.[4,11]

Example 3. A Gerber beam (*Gerberbalken*) with total length $4L$ has a hinge connection at C and constant flexural rigidity EI in its segments ABC and CDE. This beam is supported and loaded with a concentrated moment M_0 ↺ at D as shown in Fig. 12. Determine *(a)* the slopes θ_B, θ_D, and θ_E at B, D, and E, respectively; *(b)* the slope $(\theta_C)_l$ just to the left of C; *(c)* the slope $(\theta_C)_r$ just to the right of C; *(d)* the deflection y_C at C; *(e)* the deflection y_D at D.

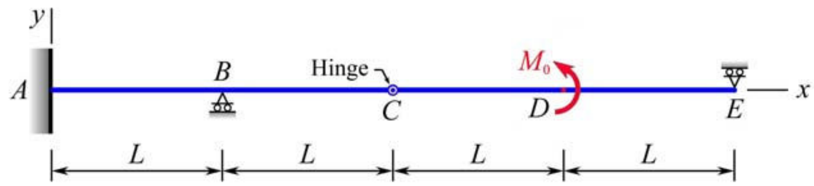

Fig. 12 A Gerber beam (actual beam)

Solution. Since the bending moment at the hinge C must be zero, we readily find that the reaction force at the right end E is $\mathbf{E}_y = M_0/(2L) \downarrow$. Clearly, this beam is statically indeterminate to the *first degree*. We note in Fig. 12 that the support conditions in the *actual beam* are as follows:

- a fixed support at A,
- a simple support (*not* at the end) at B,
- an *unsupported* hinge at C,
- a simple support at the end E.

In accordance with **rules 1, 2, 3, 5, 6,** and **7** in Section II, we draw in Fig. 13 the *conjugate beam* for the actual beam in Fig. 12. Notice in Fig. 13 that the loading diagram showing the distributed *elastic weight* on the conjugate beam is drawn *by parts*.

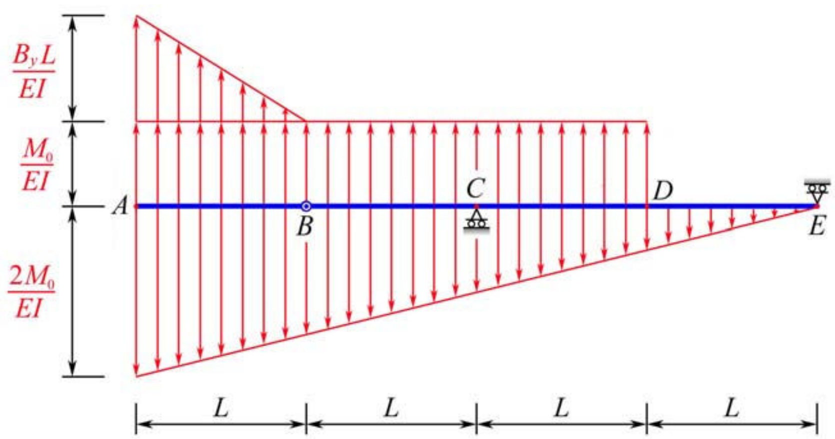

Fig. 13 Conjugate beam (additional beam) for the Gerber beam in Fig. 12

Furthermore, notice in Fig. 13 that, by **rules 3, 5, 6,** and **7** in Section II, the support conditions in the *conjugate beam* are as follows:

- a free end at A,
- an *unsupported* hinge at B,
- a simple support at C,
- a simple support at the end E.

To clearly show how this seemingly complex problem is solved, we next draw in Fig. 14 the free-body diagram for the *conjugate beam*, where the unknowns are B_y, C_y^c, and E_y^c.

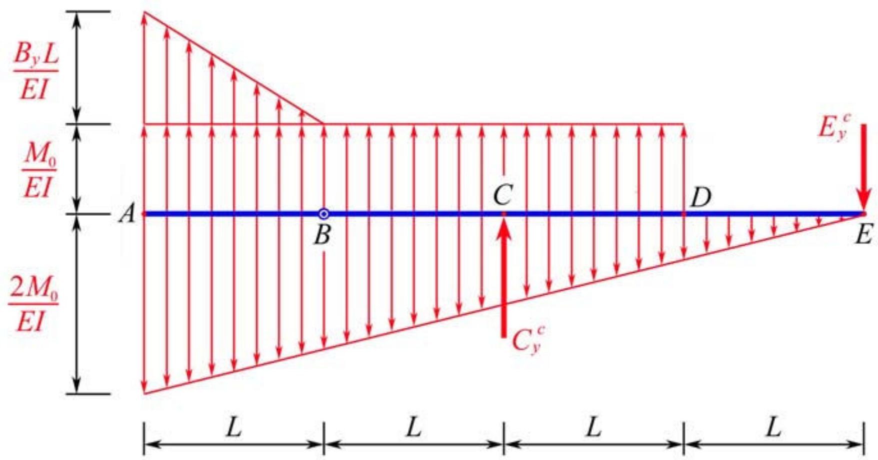

Fig. 14 Free-body diagram for the conjugate beam in Fig. 13

Note in Fig. 14 that the "bending moment" M_B^c at the hinge B must be zero. By **rule 8** in Section II, the conjugate beam (hence its free body) is in static **equilibrium**. These conditions allow us to write the equations and solutions for the unknowns B_y, C_y^c, and E_y^c as follows:

$$M_B^c = 0: \qquad \frac{2L}{3}\cdot\frac{B_y L^2}{2EI} + \frac{L}{2}\cdot\frac{M_0 L}{EI} - \frac{L}{2}\cdot\frac{3M_0 L}{2EI} - \frac{2L}{3}\cdot\frac{M_0 L}{4EI} = 0 \qquad B_y = \frac{5M_0}{4L}$$

$$+\circlearrowleft \Sigma M_E^c = 0:$$

$$-2LC_y^c - \left(\frac{2L}{3} + 3L\right)\cdot\frac{L}{2}\cdot\frac{B_y L}{EI} - \frac{5L}{2}\cdot 3L\cdot\frac{M_0}{EI} + \frac{8L}{3}\cdot\frac{4L}{2}\cdot\frac{2M_0}{EI} = 0 \qquad C_y^c = \frac{7M_0 L}{16EI}$$

$$+\uparrow \Sigma F_y^c = 0: \qquad C_y^c - E_y^c - \frac{4L}{2}\cdot\frac{2M_0}{EI} + \frac{3M_0 L}{EI} + \frac{B_y L^2}{2EI} = 0 \qquad E_y^c = \frac{M_0 L}{16EI}$$

Applying **rule 9** in Section II and referring to Fig. 14, we write

$$\theta_B = V_B^c = -\frac{L}{2}\cdot\left(\frac{2M_0}{EI} + \frac{3M_0}{2EI}\right) + \frac{M_0 L}{EI} + \frac{L^2}{2EI}\cdot\frac{5M_0}{4L} \qquad \theta_B = -\frac{M_0 L}{8EI}$$

$$\theta_D = V_D^c = E_y^c + \frac{L}{2}\cdot\frac{M_0}{2EI} \qquad \theta_D = \frac{5M_0 L}{16EI}$$

$$\theta_E = V_E^c = E_y^c \qquad \theta_E = \frac{M_0 L}{16EI}$$

$$(\theta_C)_l = (V_C^c)_l = \frac{B_y L^2}{2EI} + \frac{2M_0 L}{EI} - \frac{2L}{2}\cdot\left(\frac{2M_0}{EI} + \frac{M_0}{EI}\right) \qquad (\theta_C)_l = -\frac{3M_0 L}{8EI}$$

$$(\theta_C)_r = (V_C^c)_r = E_y^c + \frac{2L}{2}\cdot\frac{M_0}{EI} - \frac{M_0}{EI} \qquad (\theta_C)_r = \frac{M_0 L}{16EI}$$

We report that

$$\theta_B = \frac{M_0 L}{8EI} \circlearrowright \qquad \theta_D = \frac{5M_0 L}{16EI} \circlearrowleft \qquad \theta_E = \frac{M_0 L}{16EI} \circlearrowleft$$

$$(\theta_C)_l = \frac{3M_0 L}{8EI} \circlearrowright \qquad (\theta_C)_r = \frac{M_0 L}{16EI} \circlearrowleft$$

Applying **rule 10** in Section II and referring to Fig. 14, we determine the deflections y_C at C and y_D at D as follows:

$$y_C = M_C^c = \frac{L}{2}\cdot\frac{M_0 L}{EI} - \frac{2L}{3}\cdot\frac{M_0 L}{EI} - 2LE_y^c \qquad y_C = -\frac{7M_0 L^2}{24EI}$$

$$y_D = M_D^c = -LE_y^c - \frac{L}{3}\cdot\frac{L}{2}\cdot\frac{M_0}{2EI} \qquad y_D = -\frac{7M_0 L^2}{48EI}$$

We report that

$$y_C = \frac{7M_0 L^2}{24EI} \downarrow \qquad y_D = \frac{7M_0 L^2}{48EI} \downarrow$$

Based on these results, we can plot in Fig. 15 the deflection of the Gerber beam in Fig. 12.

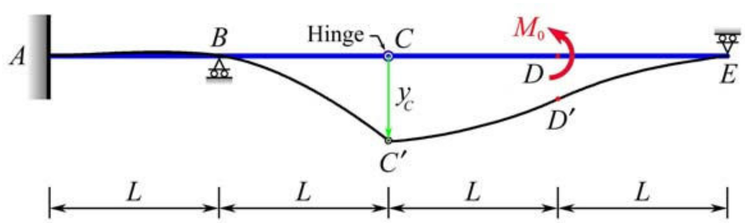

Fig. 15 Deflection of the Gerber beam in Fig. 12

Remark: *Nine* of the *ten* guiding rules have been used in this example. The results have been verified by the author to be *in agreement* with results obtained using *moment-area theorems*.[11]

Example 4. A fix-ended beam AB with length L and constant flexural rigidity EI is acted on by a concentrated moment $M_0 \circlearrowleft$ at its midpoint C, and its right fixed end B is forced to shift upward by an amount δ, without rotation, as shown in Fig. 16. Determine *(a)* the vertical reaction force $\mathbf{A}_y$ and the reaction moment $\mathbf{M}_A$ at A, *(b)* the vertical reaction force $\mathbf{B}_y$ and the reaction moment $\mathbf{M}_B$ at B.

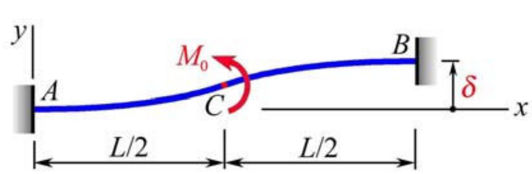

Fig. 16 Relative vertical shifting of supports in a loaded beam (actual beam)

Solution. The free-body diagram for this beam may be drawn as shown in Fig. 17, where the unknowns are A_y, M_A, B_y, and M_B. Clearly, this beam is statically indeterminate to the *second degree*.

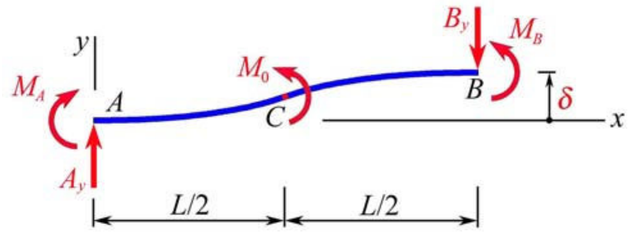

Fig. 17 Free-body diagram for the beam in Fig. 16

For equilibrium of the actual beam, we refer to Fig. 17 to write

$$+\uparrow \Sigma F_y = 0: \qquad A_y - B_y = 0 \qquad (a)$$

$$+\circlearrowleft \Sigma M_B = 0: \qquad -M_A - LA_y + M_0 + M_B = 0 \qquad (b)$$

The beam in Fig. 16 has fixed ends at A and B, but the fixed end B is forced to move upward, without rotation, an amount of δ. This means that the conjugate beam for the actual beam must, by **rules 1**, **2**, **3**, and **10** in Section II, have free ends at A and B **plus** a *counterclockwise* "bending moment" of magnitude δ acting at B, as shown in Fig. 18, where the distributed *elastic weight* on the conjugate beam is drawn *by parts*.

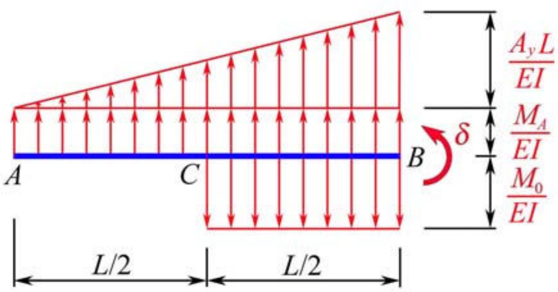

Fig. 18 Conjugate beam (additional beam) for the beam in Fig. 16

The reason we apply **rule 10** in Section II to impose a *counterclockwise* "bending moment" of magnitude δ acting at B in Fig. 18 is to take into account the extraordinary boundary condition of *upward* displacement of the fixed end B, without rotation, of the actual beam. [If an actual beam should have a specified slope (*or* rotation) θ_0 at any point, **rule 9** in Section II would, of course, be applied to impose a "shearing force" θ_0 at that point of the conjugate beam.]

By **rule 8** in Section II, the free body of the conjugate beam in Fig. 18 is in static **equilibrium** – *without* any support in space. We refer to this figure to get the *two* needed additional equations:

$$+\uparrow \Sigma F_y^c = 0: \qquad \frac{A_y L^2}{2EI} + \frac{M_A L}{EI} - \frac{M_0 L}{2EI} = 0 \qquad (c)$$

$$+\circlearrowleft \Sigma M_B^c = 0: \qquad -\frac{L}{3} \cdot \frac{A_y L^2}{2EI} - \frac{L}{2} \cdot \frac{M_A L}{EI} + \frac{L}{4} \cdot \frac{M_0 L}{2EI} + \delta = 0 \qquad (d)$$

Solving the preceding Eqs. *(a)* through *(d)* simultaneously, we obtain

$$A_y = B_y = -\frac{24EI\delta - 3M_0 L^2}{2L^3} \qquad M_A = -M_B = \frac{24EI\delta - M_0 L^2}{4L^2}$$

From these results and the *assumed* directions of forces and moments in Fig. 17, we report that

$$\mathbf{A}_y = \frac{24EI\delta - 3M_0L^2}{2L^3} \downarrow \qquad \mathbf{M}_A = \frac{24EI\delta - M_0L^2}{4L^2} \circlearrowright$$

$$\mathbf{B}_y = \frac{24EI\delta - 3M_0L^2}{2L^3} \uparrow \qquad \mathbf{M}_B = \frac{24EI\delta - M_0L^2}{4L^2} \circlearrowright$$

Remark: Equations *(c)* and *(d)* in this example can be obtained by imposing the conditions $\theta_{A/B} = 0$ (for the relative angle between the two tangents drawn at points A and B of the beam) and $t_{B/A} = \delta$ (for the tangential deviation of point B with respect to the tangent drawn at point A of the beam), respectively, in the method using *moment-area theorems*.[11] Clearly, the *same* results in this example are what will be obtained in the method using *moment-area theorems*.

Example 5. A beam AB with total length $2L$ and constant flexural rigidity EI is supported on a single simple support at its midpoint C as shown in Fig. 19. The beam is in neutral equilibrium because it carries a concentrated moment $M_0 \circlearrowleft$ at A and a concentrated force $P \downarrow$ at B, where $P = M_0/L$. Determine *(a)* the slope θ_A and deflection y_A at A, *(b)* the slope θ_B and deflection y_B at B, *(c)* the slope θ_C at C.

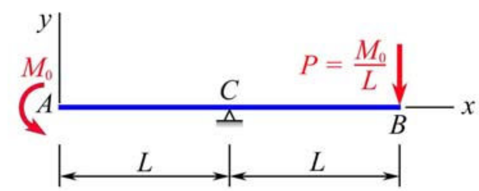

Fig. 19 Actual beam — in neutral equilibrium — (repeat of Fig. 1)

Solution. This example employs the beam in Fig. 1, which was used earlier for introductory purposes. The problem in this example cannot be tackled by any method other than the conjugate beam method. This beam manifests *three* support conditions (i.e., *free end* at A, *simple support* at C, and *free end* at B), which are sufficient to allow a corresponding conjugate beam to be constructed as shown in Fig. 20, which was shown as Fig. 2 earlier for introductory purposes.

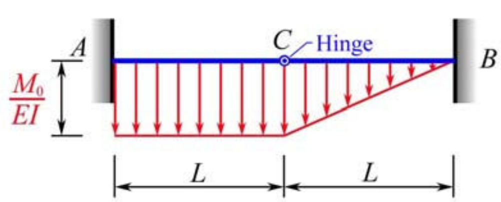

Fig. 20 Conjugate beam — for the beam in Fig. 19 — (repeat of Fig. 2)

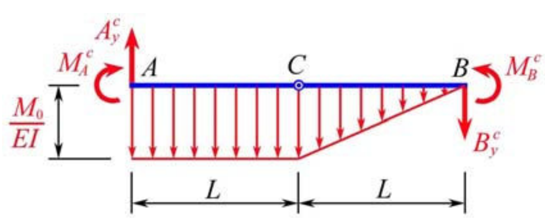

Fig. 21 Free-body diagram for the conjugate beam in Fig. 20

To clearly show how this seemingly baffling problem is solved, we draw in Fig. 21 the free-body diagram for the *conjugate beam*, where the unknowns are A_y^c, M_A^c, B_y^c, and M_B^c.

By **rule 8** in Section II, the free body in Fig. 21 is in static **equilibrium**. Moreover, the "bending moment" at the hinge C in Fig. 21 must be zero. These conditions allow us to write

$+\uparrow \Sigma F_y^c = 0$, for the *entire* conjugate beam ACB in Fig. 21:

$$A_y^c - B_y^c - \frac{M_0 L}{EI} - \frac{M_0 L}{2\,EI} = 0 \tag{a}$$

$+\circlearrowleft \Sigma M_C^c = 0$, for just segment AC — the *left* segment of the conjugate beam in Fig. 21:

$$-M_A^c - L\,A_y^c + \frac{L}{2}\cdot\frac{M_0 L}{EI} = 0 \tag{b}$$

$+\circlearrowleft \Sigma M_C^c = 0$, for just segment CB — the *right* segment of the conjugate beam in Fig. 21:

$$M_B^c - L\,B_y^c - \frac{L}{3}\cdot\frac{M_0 L}{2EI} = 0 \tag{c}$$

The above *three* equations contain *four* unknowns: A_y^c, M_A^c, B_y^c, and M_B^c. Thus, we are faced with a problem involving a conjugate beam that is statically indeterminate to the *first degree*. The statical indeterminacy of the conjugate beam in Fig. 20 can, of course, be resolved by using any of the established methods.

Let us choose to employ the conjugate beam method further to generate the needed additional equation to go with the preceding three equations *(a)*, *(b)*, and *(c)* for solving the problem in this example. For simplicity, the "flexural rigidity" of each segment of the conjugate beam in Fig. 20 may be taken as equal to 1 unit. By drawing the "elastic weight" *by parts*, we construct in Fig. 22 the "conjugate beam" for the conjugate beam in Fig. 20. Note that such a "conjugate beam" as shown in Fig. 22 has free ends at A and B and a single simple support at its midpoint C.

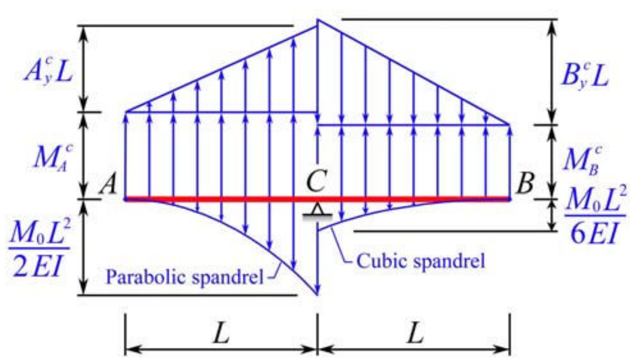

Fig. 22 "Conjugate beam" for the conjugate beam in Fig. 20

By **rule 8** in Section II, the "conjugate beam" in Fig. 22 must be in static **equilibrium**. Similar to the original actual beam in Fig. 19, the "conjugate beam" in Fig. 22 turns out to be *also* in neutral equilibrium. Using the superscripts ^{cc} to refer to the "conjugate beam" for the conjugate beam and referring to Fig. 22, we write

$+\circlearrowleft \Sigma M_C^{cc} = 0$:

$$-\frac{L}{2}\cdot M_A^c L - \frac{L}{3}\cdot\frac{A_y^c L^2}{2} + \frac{L}{4}\cdot\frac{M_0 L^3}{6EI} - \frac{L}{3}\cdot\frac{B_y^c L^2}{2} + \frac{L}{2}\cdot M_B^c L - \frac{L}{5}\cdot\left(\frac{L}{4}\cdot\frac{M_0 L^2}{6EI}\right) = 0 \qquad (d)$$

Note that Eq. *(d)* is the additional equation needed to go with the preceding Eqs. *(a)*, *(b)*, and *(c)* to resolve the statical indeterminacy mentioned above.

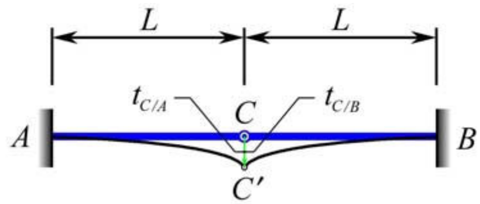

Fig. 23 "Deflection" of the conjugate beam in Fig. 20

The *raison d'être* for the above obtained Eq. *(d)* may briefly be examined. When the conjugate beam under *elastic weight* in Fig. 20 deflects, it will adopt a shape as illustrated in Fig. 23. Those familiar with the method using moment-area theorems[11] will readily perceive that the tangential deviation $t_{C/B}$ of point *C* with respect to the tangent drawn at point *B* is equal to the tangential deviation $t_{C/A}$ of point *C* with respect to the tangent drawn at point *A*; i.e.,

$$t_{C/B} = t_{C/A}$$

According to the **second moment-area theorem**,[11] the tangential deviation $t_{C/B}$ is equal to the first moment, taken *counterclockwise* about point *C*, of the "elastic weight" between points *B* and *C* in Fig. 22. Meantime, the tangential deviation $t_{C/A}$ is equal to the first moment, taken *clockwise* about point *C*, of the "elastic weight" between points *A* and *C* in Fig. 22. By transposing terms in the above equation, we see that

$$-t_{C/A} + t_{C/B} = 0$$

By carrying out the **second moment-area theorem** for terms in this equation, we see that this equation leads to the *same* equation as Eq. *(d)* above. Thus, applying the conjugate beam method to further study the statically indeterminate conjugate beam in Fig. 20 is sound and well.

Solving the preceding Eqs. *(a)* through *(d)* simultaneously for the four unknowns in them, we get

$$A_y^c = \frac{19\,M_0 L}{20EI} \qquad B_y^c = -\frac{11\,M_0 L}{20EI}$$

$$M_A^c = -\frac{9\,M_0 L^2}{20EI} \qquad M_B^c = -\frac{23\,M_0 L^2}{60EI}$$

Using these results and applying **rules 9** and **10** in Section II, we can refer to Fig. 21 and write

$$\theta_A = V_A^c = A_y^c = \frac{19 M_0 L}{20EI} \qquad \theta_B = V_B^c = B_y^c = -\frac{11\,M_0 L}{20EI}$$

$$\theta_C = V_C^c = A_y^c - \frac{M_0 L}{EI} = -\frac{M_0 L}{20EI}$$

$$y_A = M_A^c = -\frac{9 M_0 L^2}{20EI} = -\frac{27 M_0 L^2}{60EI} \qquad y_B = M_B^c = -\frac{23 M_0 L^2}{60EI}$$

For the loaded beam in neutral equilibrium in Figs. 1 and 19, we report that

$$\theta_A = \frac{19 M_0 L}{20EI} \circlearrowleft \qquad \theta_B = \frac{11 M_0 L}{20EI} \circlearrowright \qquad \theta_C = \frac{M_0 L}{20EI} \circlearrowright$$

$$y_A = \frac{27 M_0 L^2}{60EI} \downarrow \qquad y_B = \frac{23 M_0 L^2}{60EI} \downarrow$$

Based on these results, we depict in Fig. 24 the deflection of the actual beam in Figs. 1 and 19.

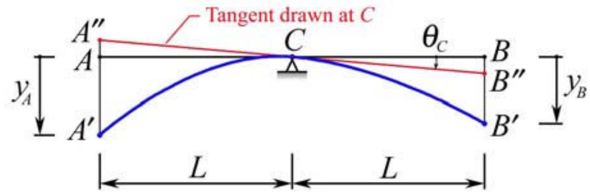

Fig. 24 Deflection of the actual beam in Figs. 1 and 19

Remark: Since the problem in this example *cannot* be solved by *any* other methods, no direct comparison of the preceding results can be made. Nonetheless, assessment of these results is possible as presented in Section IV.

IV. Assessment of Results Obtained in Example 5

Let us refer to both Fig. 19 and Fig. 24. Since we have obtained the slope θ_C for the tangent $A''CB''$ drawn at C in Fig. 24, we may perform an analytical check of the solutions by regarding the deflected shape of this beam AB as the elastic curve of two cantilever beams:

- a cantilever beam CA'' with length L, fixed at C, which is deflected from CA'' to CA' by a concentrated moment $M_0 \circlearrowleft$ at A'';
- a cantilever beam CB'' with length L, fixed at C, which is deflected from CB'' to CB' by a concentrated force $P \downarrow$ at B'', where $P = M_0/L$.

For ease of reference, we repeat the beam and the results as shown below.

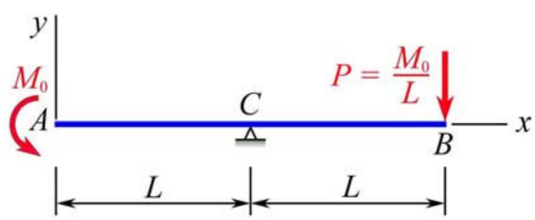

Fig. 19 Actual beam — in neutral equilibrium — (repeat of Fig. 1)

$$\theta_A = \frac{19 M_0 L}{20EI} \circlearrowleft \qquad \theta_B = \frac{11 M_0 L}{20EI} \circlearrowright \qquad \theta_C = \frac{M_0 L}{20EI} \circlearrowright \qquad y_A = \frac{27 M_0 L^2}{60EI} \downarrow \qquad y_B = \frac{23 M_0 L^2}{60EI} \downarrow$$

From the geometry in Fig. 24 and the results as shown, we find the following:

$$\overline{AA''} = \overline{BB''} = L\left|\theta_C\right| = \frac{M_0 L^2}{20EI}$$

$$\overline{A''A'} = \overline{AA''} + \left|y_A\right| = \frac{M_0 L^2}{20\,EI} + \frac{27\,M_0 L^2}{60\,EI} = \frac{M_0 L^2}{2\,EI}$$

$$\theta_{A/C} = \theta_A - \theta_C = \frac{19 M_0 L}{20EI} - \left(-\frac{M_0 L}{20EI}\right) = \frac{M_0 L}{EI}$$

$$\overline{B''B'} = \left|y_B\right| - \overline{BB''} = \frac{23 M_0 L^2}{60EI} - \frac{M_0 L^2}{20EI} = \frac{M_0 L^2}{3EI} = \frac{(PL)L^2}{3EI} = \frac{PL^3}{3EI}$$

$$\theta_{B/C} = \theta_B - \theta_C = -\frac{11\,M_0 L}{20EI} - \left(-\frac{M_0 L}{20EI}\right) = -\frac{M_0 L}{2EI} = -\frac{(PL)L}{2EI} = -\frac{PL^2}{2EI}$$

We have

$$\overline{A''A'} = \frac{M_0 L^2}{2\,EI} \qquad \theta_{A/C} = \frac{M_0 L}{EI} \qquad \overline{B''B'} = \frac{PL^3}{3EI} \qquad \theta_{B/C} = -\frac{PL^2}{2EI}$$

We note that the above values for $\overline{A''A'}$, $\theta_{A/C}$, $\overline{B''B'}$, and $\theta_{B/C}$ are all *in agreement* with those found in a table *or* an appendix of textbooks[4,11] for the deflection and slope of the free end of a cantilever beam loaded at its free end with *(a)* a concentrated moment $\mathbf{M}_0$, *(b)* a concentrated force **P**, respectively.

Note that no rigorous *experimental* results for deflections of beams in neutral equilibrium are readily available. In the absence of any available results for direct comparison, the foregoing agreeable assessment may be taken as a "pat on the back" for the efforts and results obtained in solving a baffling problem in Example 5.

V. Concluding Remarks

This paper is written to share with fellow engineering educators the insights, highlights, and *five* detailed illustrative examples for teaching the conjugate beam method. It is pointed out that more *support* conditions than *boundary* conditions are usually known for beams in neutral equilibrium. The conjugate beam method can readily handle the following five basic support conditions (*or* types): *(i)* fixed end, *(ii)* free end, *(iii)* simple support at the end, *(iv)* simple support *not* at the end, and *(v) unsupported* hinge. Furthermore, **rules 9** and **10** in Section II can readily be applied to address extraordinary boundary conditions, where s*pecific values* for slopes and deflections are present or stipulated in the problem. This is a versatile feature inherent in the guiding rules for the conjugate beam method, as illustrated in Example 4.

Besides being able to correctly find solutions for complex as well as simple problems of deflections of beams, as illustrated in Examples 1 through 4, the conjugate beam method stands out as the *only* method that is able to pursue and yield the solution for the deflection of a loaded beam in neutral equilibrium, as illustrated in Example 5. This method is unique and outstanding. The

root cause contributing to this rather unusual scenario lies in the use of **support conditions** in the conjugate beam method *versus* the use of **boundary conditions** in all other methods.

In using the conjugate beam method, more *statics skills* and very little explicit *calculus skills* are usually needed. When the support conditions of a beam are properly recognized and taken into account by using the guiding rules in the conjugate beam method, the **boundary conditions** will, of course, be **satisfied** automatically! In other words, all solutions generated by the conjugate beam method have satisfied the boundary conditions in the beginning stages of the solutions. The conjugate beam method has been demonstrated in this paper to do a job *as good as* or *better than* other established methods in determining deflections of beams.

For years, the author has taught the *conjugate beam method*, in addition to the established methods using *double integration* and *moment-area theorems*, in his own class of mechanics of materials at his home institution, where the students have been provided with the **10 Guiding Rules in Conjugate Beam Method** on a single page. Overwhelmingly, his students favor the conjugate beam method over the other methods in determining deflections of beams.

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