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Determining Deflections of Elastic Beams: What Can the Conjugate Beam Method Do That All Others Cannot?*

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An elastic beam on a simple support can be in neutral equilibrium under a variety of loading conditions. Is it possible to ascertain the deflections of such a beam? The answer is: yes—by the conjugate beam method propounded by Westergaard in 1921, and no—by all other methods. It is recognized that support conditions, rather than boundary conditions, are what the conjugate beam method requires in finding deflections of loaded beams; and more support conditions than boundary conditions are usually known for beams in neutral equilibrium. The objective of this paper is to share with engineering educators the pedagogy of the conjugate beam method and the solution for the deflected configuration of a loaded elastic beam in neutral equilibrium by this method. The feasibility of obtaining such a solution via this method is unmatched by other methods. The conjugate beam method, missing in most current textbooks in mechanics of materials, is as good as (or even better than) other methods when it comes to the analysis of deflections of beams. Once well revived in textbooks, or otherwise, for teaching and learning, this method is expected to significantly impact the favored way beam deflections are analyzed.

Keywords: beam; slope; deflection; neutral equilibrium; conjugate beam method

NOMENCLATURE

EI	Flexural rigidity of the given beam
L	Parameter to indicate a length
M/EI	Elastic weight (i.e., a fictitious load per unit length) on the conjugate beam
P	Parameter to indicate a concentrated force
A_y^c, B_y^c	Vertical reaction forces on the conjugate beam at points A, B
M_A^c, M_B^c	Bending moments in the conjugate beam at points A, B
V_A^c, V_D^c	Vertical shear forces in the conjugate beam at points A, D
θ_A, θ_D	Slopes of the given beam at points A, D
θ_{BL}, θ_{BR}	Slopes of beam just to the left ($_L$) of point B, just to the right ($_R$) of point B
y_A, y_D	Deflections of the given beam at points A, D
$t_{C/A}$	Tangential deviation (vertical displacement) of point C with respect to the tangent drawn at point A of the deflected given beam (in moment-area theorems)
ΣF_y^c	Summation of elastic weights, in the vertical direction, on the conjugate beam
ΣM_C^c	Summation of moments, of elastic weights, about point C on the conjugate beam
ΣM_C^{cc}	Summation of moments, of "elastic weights," about point C on the "conjugate beam" for a conjugate beam.

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1. INTRODUCTION

A BEAM IS IN NEUTRAL EQUILIBRIUM if the force system acting on the beam is statically balanced and the potential energy of the beam in the neighborhood of its equilibrium configuration is constant.

The given beam in Fig. 1 has a flexural rigidity EI and is in neutral equilibrium. Being elastic, this beam will adopt a deflected shape. Question**:** What method can be used to ascertain the configuration of the deflected shape of this beam? Answer**:** The conjugate beam method [1] can, all others cannot. The other methods [2–10] for determining deflections of beams include:

- method of integration (with *or* without the use of singularity functions);
- method of superposition;
- method using moment-area theorems;
- method using Castigliano's theorem;
- method of model formulas.

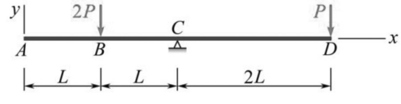

Fig. 1. Beam in neutral equilibrium on a simple support.

The latter methods all expect a beam to have sufficiently well-defined boundary conditions for use in seeking a unique solution for the configuration of the deflected beam. The beam in Fig. 1 manifests only one known boundary condition (i.e., the deflection at the hinge support C is zero), which is simply insufficient to allow any of the other methods to settle on a unique solution.

However, the beam in Fig. 1 is not a puzzle to the conjugate beam method. This beam manifests three support conditions (i.e., free end at A, simple support at C, and free end at D), which are sufficient to allow a corresponding conjugate beam to be constructed and employed to ascertain the configuration of the deflected shape of this beam. This beam problem will be solved in Section 2.

1.1 Pedagogy of the conjugate beam method

The conjugate beam method is actually a natural extension of the moment-area theorems. It is an elegant, efficient, and powerful method published by Westergaard [1] some nine decades ago, although some considered Mohr (1868) and Breslau (1865) to have prior influences. Elementary presentation of this method did appear in early textbooks in mechanics of materials [2, 3]. For reasons unknown, this method is missing in most such current textbooks. The pedagogy of the conjugate beam method lies in teaching and applying the rules in this method [1, 11]. These rules are summarized as follows:

- **Rule 1**: The conjugate beam and the given beam are of the same *length*.
- **Rule 2**: The load on the conjugate beam is the *elastic weight*, which is the bending moment M in the given beam divided by the flexural rigidity EI of the given beam. (This elastic weight is taken to act upward if the bending moment is positive—to cause top fiber in compression—in beam convention.)

For each existing support condition of the given beam, there is a corresponding support condition for the conjugate beam. The correspondence is given by rules 3 through 7 as follows:

	Existing support condition in the **given beam:**	Corresponding support condition in the **conjugate beam:**
Rule 3:	Fixed end	Free end
Rule 4:	Free end	Fixed end
Rule 5:	Simple support at the end	Simple support at the end
Rule 6:	Simple support *not* at the end	Unsupported hinge
Rule 7:	Unsupported hinge	Simple support

- **Rule 8**: The conjugate beam is in static equilibrium.
- **Rule 9**: The slope of the given beam at any cross section is given by the "shear force" at that cross section of the conjugate beam. (This *slope* is positive, or counterclockwise, if the "shear force" is positive—tending to rotate the beam element clockwise—in beam convention.)
- **Rule 10**: The deflection of the given beam at any point is given by the "bending moment" at that point of the conjugate beam. (This *deflection* is upward if the "bending moment" is positive—tending to cause the top fiber in compression—in beam convention.)

1.2 Illustration of the pedagogy

A combined beam, with a constant flexural rigidity EI, fixed supports at its ends A and D, a hinge connection at B, and carrying a concentrated force $\mathbf{P}$ at C, is shown in Fig. 2. Determine (a) the vertical reaction force $\mathbf{A}_y$ and the reaction moment $\mathbf{M}_A$ at A, (b) the deflection y_B of the hinge at B, (c) the slopes θ_{BL} and θ_{BR} just to the left and just to the right of the hinge at B, respectively, and (d) the slope θ_C and the deflection y_C at C.

Solution: We note that the beam given in Fig. 2 is statically indeterminate to the first degree. Using $\mathbf{A}_y$ as the redundant unknown, we may assume that the reaction force and reaction moment at A are as shown in Fig. 3. Drawing the moment-diagram by parts, we may construct the corresponding conjugate beam as shown in Fig. 4.

Note in Figs. 2 and 4 the following key points:

- The conjugate beam and the given beam are of the same length;

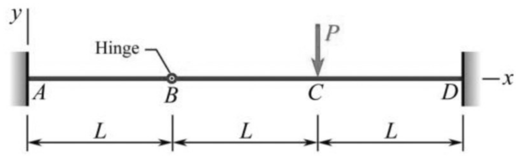

Fig. 2. Statically indeterminate beam with a hinge connection.

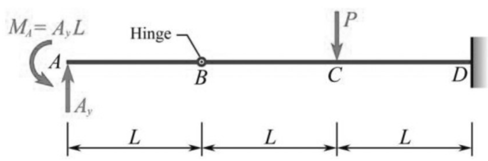

Fig. 3. Reactions at end A of the beam in Fig. 2.

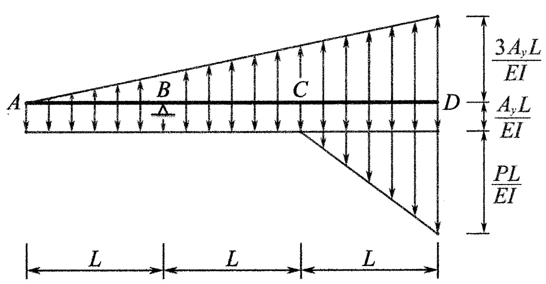

Fig. 4. Conjugate beam for the beam in Figs. 2 and 3.

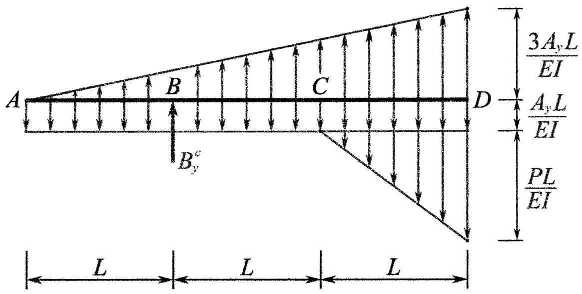

Fig. 5. Free-body diagram of the conjugate beam in Fig. 4.

- The fixed ends at A and D in the given beam change to free ends at A and D in the conjugate beam;
- The unsupported hinge at B in the given beam changes to a simple support at B in the conjugate beam;
- The elastic weight acting on the conjugate beam comes from the bending moment M in the given beam divided by the flexural rigidity EI of the given beam.

The conjugate beam in Fig. 4 and the free body of the conjugate beam in Fig. 5 are in static equilibrium. Referring to the entire free-body diagram in Fig. 5, we write $+\circlearrowleft\ \Sigma M_B^c = 0$:

$$L \cdot \frac{3L}{2} \cdot \frac{3A_yL}{EI} - \frac{L}{2} \cdot 3L \cdot \frac{A_yL}{EI} - \left(L + \frac{2L}{3}\right) \cdot \frac{L}{2} \cdot \frac{PL}{EI} = 0 \qquad (1)$$

This equation yields

$$A_y = \frac{5P}{18} \qquad M_A = A_yL = \frac{5PL}{18} \qquad (2)$$

We report that

$$\mathbf{A}_y = \frac{5P}{18}\uparrow \qquad \mathbf{M}_A = \frac{5PL}{18}\ \circlearrowleft$$

Referring to Fig. 5, we write $+\uparrow \Sigma F_y^c = 0$:

$$B_y^c + \frac{3L}{2} \cdot \frac{3A_yL}{EI} - 3L \cdot \frac{A_yL}{EI} - \frac{L}{2} \cdot \frac{PL}{EI} = 0 \qquad (3)$$

This equation yields

$$B_y^c = \frac{PL^2}{12EI} \qquad (4)$$

Using the above obtained values, referring to Fig. 5, and applying the rules in the conjugate beam method in Section 1.1, we may compute and report the requested quantities as follows:

$$y_B = M_B^c = \frac{L}{3} \cdot \frac{L}{2} \cdot \frac{A_yL}{EI} - \frac{L}{2} \cdot \frac{A_yL^2}{EI} = -\frac{5PL^3}{54EI}$$

$$y_B = \frac{5PL^3}{54EI}\downarrow$$

$$\theta_{BL} = V_{BL}^c = \frac{L}{2} \cdot \frac{A_yL}{EI} - \frac{A_yL^2}{EI} = -\frac{5PL^2}{36EI}$$

$$\theta_{BL} = \frac{5PL^2}{36EI}\ \circlearrowright$$

$$\theta_{BR} = V_{BR}^c = V_{BL}^c + B_y^c = -\frac{2PL^2}{36EI}$$

$$\theta_{BR} = \frac{2PL^2}{36EI}\ \circlearrowright$$

$$\theta_C = V_C^c = \frac{2L}{2} \cdot \frac{2A_yL}{EI} + B_y^c - 2L \cdot \frac{A_yL}{EI} = \frac{PL^2}{12EI}$$

$$\theta_C = \frac{PL^2}{12EI}\ \circlearrowleft$$

$$y_C = M_C^c = \frac{2L}{3} \cdot \frac{2L}{2} \cdot \frac{2A_yL}{EI} + LB_y^c - L \cdot \frac{2A_yL^2}{EI} = -\frac{11PL^3}{108EI}$$

$$y_C = \frac{11PL^3}{108EI}\downarrow$$

Based on the preceding solutions, deflections of the beam in Fig. 2 are depicted in Fig. 6.

2. DEFLECTIONS OF A BEAM IN NEUTRAL EQUILIBRIUM: CONJUGATE BEAM METHOD

The slopes and deflections of the beam in neutral equilibrium in Fig. 1 are puzzles to all methods, except the conjugate beam method. The deflections of this beam may now be investigated via its conjugate beam as shown in Fig. 7, which is constructed according to the rules as summarized in Section 1.1.

For solving the conjugate beam in Fig. 7, we

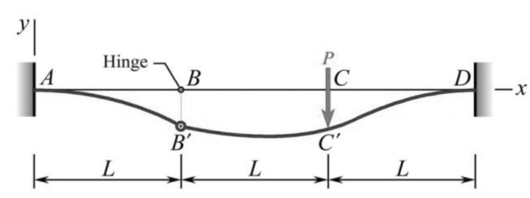

Fig. 6. Obtained configuration of deflections of the beam in Fig. 2.

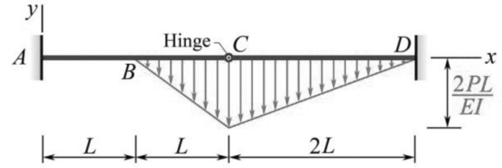

Fig. 7. Conjugate beam for the beam in Fig. 1.

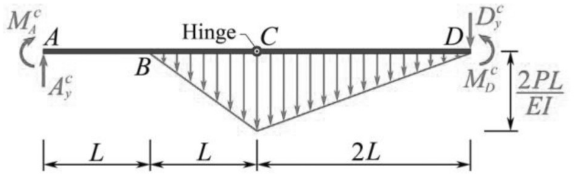

Fig. 8. Free-body diagram for the conjugate beam in Fig. 7.

draw its free-body diagram as shown in Fig. 8. We note that the free body in Fig. 8 is in static equilibrium. Moreover, the "bending moment" at the hinge C in Fig. 8 must be zero. These conditions allow us to write $+\uparrow \Sigma F_y^c = 0$, for the *entire* conjugate beam $ABCD$ in Fig. 8:

$$A_y^c - D_y^c - \frac{3L}{2}\cdot\frac{2PL}{EI} = 0 \qquad (5)$$

$+\circlearrowleft \Sigma M_C^c = 0$, for just segment ABC — the *left* segment of the conjugate beam in Fig. 8:

$$-M_A^c - 2L\,A_y^c + \frac{L}{3}\cdot\frac{L}{2}\cdot\frac{2PL}{EI} = 0 \qquad (6)$$

$+\circlearrowleft \Sigma M_C^c = 0$, for just segment CD — the *right* segment of the conjugate beam in Fig. 8:

$$M_D^c - 2L\,D_y^c - \frac{2L}{3}\cdot\frac{2L}{2}\cdot\frac{2PL}{EI} = 0 \qquad (7)$$

Equations (5), (6), and (7) contain four unknowns: A_y^c, M_A^c, D_y^c, and M_D^c. Thus, we are faced with a problem involving a conjugate beam that is statically indeterminate to the first degree. The statical indeterminacy of the conjugate beam in Figs. 7 and 8 can, of course, be resolved by using any of the established methods.

Let us employ the conjugate beam method further to obtain the needed additional equation to go with the preceding Eqs. (5), (6), and (7) for solving the problem. For simplicity, the "flexural rigidity" of each segment of the conjugate beam in Figs. 7 and 8 may be taken as equal to 1 unit. By drawing the "elastic weight" by parts, we construct the "conjugate beam" in Fig. 9 for the conjugate beam in Figs. 7 and 8. Note that such a "conjugate beam" as shown in Fig. 9 has free ends at A and D and a simple support at its midpoint C.

The "conjugate beam" in Fig. 9 is in static equilibrium. Similar to the original given beam in Fig. 1, the "conjugate beam" in Fig. 9 turns out to be also in neutral equilibrium. Using the superscripts ^{cc} to refer to the "conjugate beam" for the conjugate beam and referring to Fig. 9, we can write $+\circlearrowleft \Sigma M_C^{cc} = 0$:

$$-\frac{2L}{2}\cdot M_A^c(2L) - \frac{2L}{3}\cdot\frac{2L}{2}(2A_y^cL) + \frac{L}{5}\cdot\left(\frac{L}{4}\cdot\frac{PL^3}{3EI}\right) - \frac{2L}{3}\cdot\left(\frac{2L}{2}\cdot 2D_y^cL\right) + \frac{2L}{2}\cdot M_D^c(2L) - \frac{2L}{5}\cdot\left(\frac{2L}{4}\cdot\frac{4PL^3}{3EI}\right) = 0 \qquad (8)$$

Equation (8) is the additional equation needed to go with the preceding Eqs. (5), (6), and (7) to resolve the statical indeterminacy mentioned above.

Justifying Eq. (8): The justification and validity of Eq. (8) can briefly be examined. When the conjugate beam under elastic weight in Figs. 7 and 8 "deflects," it will adopt a shape as illustrated in Fig. 10. According to the second moment-area theorem [9], the tangential deviation $t_{C/D}$ indicated in Fig. 10 is equal to the first moment, $+\circlearrowleft(M_C^{cc})_{CD}$, taken counterclockwise about point C, of the "elastic weight" between points C and D in Fig. 9. Meanwhile, the tangential deviation $t_{C/A}$ indicated in Fig. 10 is equal to the first moment, $+\circlearrowright(M_C^{cc})_{AC}$, taken clockwise about point C, of the "elastic weight" between points A and C in Fig. 9. Thus, we have

$$t_{C/D} = +\circlearrowleft(M_C^{cc})_{CD} \quad t_{C/A} = +\circlearrowright(M_C^{cc})_{AC} \qquad (9)$$

By inspection, we see in Fig. 10 that

$$t_{C/D} = t_{C/A} \quad t_{C/D} - t_{C/A} = 0 \qquad (10)$$

Thus, we write

$$\begin{aligned} t_{C/D} - t_{C/A} &= \left[+\circlearrowleft(M_C^{cc})_{CD}\right] - \left[+\circlearrowright(M_C^{cc})_{AC}\right] \\ &= \left[+\circlearrowleft(M_C^{cc})_{CD}\right] + \left[+\circlearrowleft(M_C^{cc})_{AC}\right] \\ &= +\circlearrowleft(M_C^{cc})_{AD} = +\circlearrowleft \Sigma M_C^{cc} = 0 \qquad (11) \end{aligned}$$

Equation (11) verifies that the indicated source equation that yields Eq. (8) is indeed true and valid in the eyes of moment-area theorems. Thus, using the "conjugate beam" in Fig. 9 to further study the statically indeterminate conjugate beam in Figs. 7 and 8 is sound and well.

Solving the preceding Eqs. (5) through (8) simultaneously for the four unknowns in them, we get

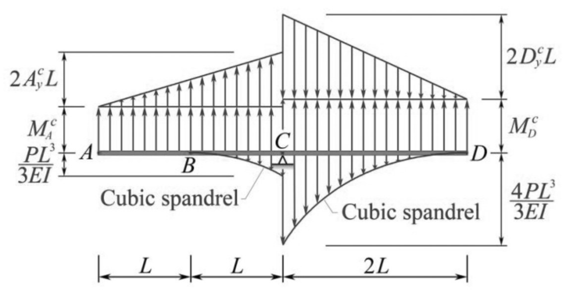

Fig. 9. "Conjugate beam" for the conjugate beam in Figs. 7 and 8.

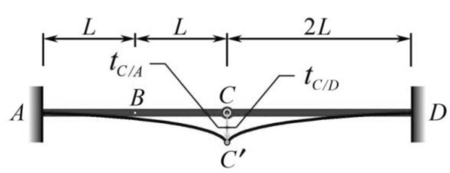

Fig. 10. "Deflection" of the conjugate beam in Figs. 7 and 8.

$$A_y^c = \frac{75PL^2}{64EI} \quad D_y^c = -\frac{117PL^2}{64EI} \tag{12}$$

$$M_A^c = -\frac{193PL^3}{96EI} \quad M_D^c = -\frac{223PL^3}{96EI} \tag{13}$$

Using the above results and applying the rules in the conjugate beam method in Section 1.1, we refer to the conjugate beam in Fig. 8 and write

$$\begin{aligned} \theta_A = \theta_B = V_A^c = A_y^c &= \frac{75PL^2}{64EI} \\ \theta_C = V_C^c = A_y^c - \frac{L}{2}\cdot\frac{2PL}{EI} &= \frac{11PL^2}{64EI} \\ \theta_D = V_D^c = D_y^c &= -\frac{117PL^2}{64EI} \end{aligned} \tag{14}$$

$$y_A = M_A^c = -\frac{193PL^3}{96EI} \tag{15}$$

$$\begin{aligned} y_B = M_B^c = M_A^c + LA_y^c &= -\frac{161PL^3}{192EI} \\ y_D = M_D^c &= -\frac{223PL^3}{96EI} \end{aligned} \tag{16}$$

For the beam in neutral equilibrium in Fig. 1, we report its *slopes* and *deflections* as follows:

$$\theta_A = \theta_B = \frac{75PL^2}{64EI} \circlearrowleft \quad \theta_C = \frac{11PL^2}{64EI} \circlearrowleft$$
$$\theta_D = \frac{117PL^2}{64EI} \circlearrowright \quad y_A = \frac{193PL^3}{96EI} \downarrow$$
$$y_B = \frac{161PL^3}{192EI} \downarrow \quad y_D = \frac{223PL^3}{96EI} \downarrow$$

Based on these solutions, deflections of the beam in Fig. 1 are depicted in Fig. 11.

3. ASSESSMENT OF OBTAINED CONFIGURATION OF DEFLECTED BEAM

Since the problem in this example cannot be solved by any other methods, no direct comparison for the obtained results can be made. Nonetheless, assessment of the obtained configuration of deflections in Fig. 11 is possible. Let us refer to both Fig.

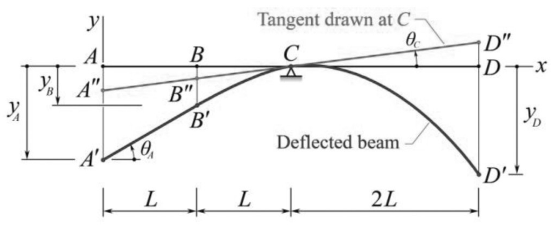

Fig. 11. Obtained configuration of deflections of the beam in Fig. 1

1 and Fig. 11. Since we have obtained the slope θ_C for the tangent $A''B''CD''$ drawn at C in Fig. 11, we may perform an analytical check of the obtained solutions by regarding the bent shape of the curved segment BCD of this beam as the elastic curve of the deflected shape of the following two beams cantilevered at C:

- Cantilever beam CB'': This beam has a length of L, cantilevered at C, and is deflected from CB'' to CB' by a concentrated force $2P \downarrow$ at B''.
- Cantilever beam CD'': This beam has a length of $2L$, cantilevered at C, and is deflected from CD'' to CD' by a concentrated force $P \downarrow$ at D''.

From the geometry in Fig. 11 and the slopes and deflections obtained in the preceding solution, we find the following:

$$\overline{BB''} = L\theta_C = \frac{11PL^3}{64EI} \quad \overline{D''D} = 2L\theta_C = \frac{11PL^3}{32EI} \tag{17}$$

$$\overline{B''B'} = |y_B| - \overline{BB''} = \frac{161PL^3}{192EI} - \frac{11PL^3}{64EI} = \frac{2PL^3}{3EI} \tag{18}$$

$$\overline{D''D'} = \overline{D''D} + |y_D| = \frac{11PL^3}{32EI} + \frac{223PL^3}{96EI} = \frac{8PL^3}{3EI} \tag{19}$$

$$\theta_{B/C} = \theta_B - \theta_C = \frac{75PL^2}{64EI} - \frac{11PL^2}{64EI} = \frac{PL^2}{EI} \tag{20}$$

$$\theta_{D/C} = \theta_D - \theta_C = -\frac{117PL^2}{64EI} - \frac{11PL^2}{64EI} = -\frac{2PL^2}{EI} \tag{21}$$

The preceding values for the several geometric quantities in Fig. 11 may equivalently be written as follows:

$$\overline{B''B'} = \frac{(2P)L^3}{3EI} \qquad \overline{D''D'} = \frac{P(2L)^3}{3EI}$$
$$\theta_{B/C} = \frac{(2P)L^2}{2EI} \qquad \theta_{D/C} = -\frac{P(2L)^2}{2EI}$$

We readily see that the above values for $\overline{B''B'}$, $\overline{D''D'}$, $\theta_{B/C}$, and $\theta_{D/C}$ are all in agreement with those found in a table *or* an appendix in the published literature [2–9] that lists the slope and deflection of the free end of a cantilever beam with flexural rigidity EI and length L, where a vertical concentrated force $P \downarrow$ acts at the free end. No doubt, the preceding obtained results are consistent with the established known results.

4. CONCLUSIONS

Conventional wisdom in the solution of a differential equation governing the deflection of a beam, or the behavior of a certain physical system, expects and requires that adequate boundary conditions be available and be satisfied before a unique solution can be obtained. Westergaard's conjugate beam method employs support conditions and hence bypasses the protocol requiring adequate boundary conditions for solving problems of beam deflections. This approach works well because boundary conditions have, in fact, been taken into account in the conjugate beam method when the support conditions are specified in the beginning stages of the solutions.

More support conditions than boundary conditions are usually known for beams in neutral equilibrium. The conjugate beam method can readily handle five basic support conditions:

- fixed end;
- free end;
- simple support at the end;
- simple support not at the end;
- unsupported hinge.

This method usually requires no explicit integration in the solution, and it requires good skills in statics in the operation. The conjugate beam method is suitable for learning by sophomores and juniors; and it has been taught, tested, and highlighted in the course MEEG 3013 Mechanics of Materials at the University of Arkansas for several years. In the analysis of beam deflections, this method is the one most frequently preferred by the students.

The conjugate beam method is unique and outstanding. It is the only analytical method that can be applied to investigate the deflection of a beam in neutral equilibrium. This method is concise and efficacious. Above all, it attests that some early ideas in engineering could be still useful today and attention should be paid to them.

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