AC 2010-88: ENRICHING STUDENTS' STUDY OF BEAM REACTIONS AND DEFLECTIONS: FROM SINGULARITY FUNCTIONS TO METHOD OF MODEL FORMULAS

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Enriching Students' Study of Beam Reactions and Deflections: From Singularity Functions to Method of Model Formulas

Abstract

Since publication of the *method of model formulas* in a recent issue of the *IJEE*,[1] there has been considerable interest in knowing a good approach to teaching this method to enrich students' study and set of skills in determining statically indeterminate reactions and deflections of elastic beams. This paper is aimed at sharing with mechanics educators an approach that can be used to effectively introduce and teach such a method.

It is a considered opinion that the *method of model formulas* be taught to students after having taught them one or more of the traditional methods. Besides enhancing the learning experience of upper class engineering students, this method can benefit practicing engineers. In particular, this method may readily serve as an independent and effective means to quickly check or assess the solutions obtained using other methods.

I. Introduction

Beams are longitudinal members subjected to transverse loads. Students usually first learn the design of beams for strength. Then they learn the determination of deflection of beams under a variety of loads. Traditional methods that are used in determining statically indeterminate reactions and deflections of elastic beams include:[2-12] method of integration (*with* or *without* the use of singularity functions), method of superposition, method using moment-area theorems, method using Castigliano's theorem, method of conjugate beam, and method of segments.

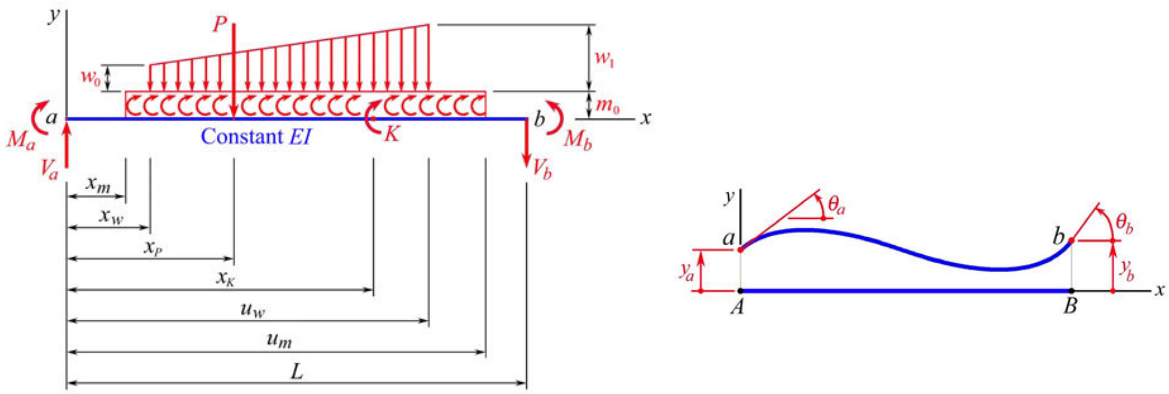
The *method of model formulas*[1] is a newly propounded method. Beginning with a general preset model loading on a beam, a set of four model formulas are established for use in this method. These formulas are expressed in terms of the following:

(a) flexural rigidity of the beam;
(b) slopes, deflections, shear forces, and bending moments at both ends of the beam;
(c) typical applied loads (concentrated force, concentrated moment, linearly distributed force, and uniformly distributed moment) somewhere on the beam.

For starters, one must know that a working **proficiency** in the rudiments of *singularity functions* is a **prerequisite** to using the *method of model formulas*. To benefit a wider readership, who may have different specialties in mechanics, and to avoid or minimize any possible misunderstanding, this paper includes summaries of the rudiments of singularity functions and the sign conventions for beams. Readers, who are familiar with these topics, may skip the summaries. An excerpt from the *method of model formulas* is needed and shown in Fig. 1, courtesy of *IJEE*.[1]

Excerpt from the **Method of Model Formulas**

Courtesy: *Int. J. Engng. Ed.*, Vol. 25, No. 1, pp. 65-74, 2009



(a) *(b)*

Positive directions of forces, moments, slopes, and deflections

$$y' = \theta_a + \frac{V_a}{2EI}x^2 + \frac{M_a}{EI}x - \frac{P}{2EI}<x-x_P>^2 + \frac{K}{EI}<x-x_K>^1 - \frac{w_0}{6EI}<x-x_w>^3 - \frac{w_1-w_0}{24EI(u_w-x_w)}<x-x_w>^4 + \frac{w_1}{6EI}<x-u_w>^3 + \frac{w_1-w_0}{24EI(u_w-x_w)}<x-u_w>^4 + \frac{m_0}{2EI}<x-x_m>^2 - \frac{m_0}{2EI}<x-u_m>^2 \tag{1}$$

$$y = y_a + \theta_a x + \frac{V_a}{6EI}x^3 + \frac{M_a}{2EI}x^2 - \frac{P}{6EI}<x-x_P>^3 + \frac{K}{2EI}<x-x_K>^2 - \frac{w_0}{24EI}<x-x_w>^4 - \frac{w_1-w_0}{120EI(u_w-x_w)}<x-x_w>^5 + \frac{w_1}{24EI}<x-u_w>^4 + \frac{w_1-w_0}{120EI(u_w-x_w)}<x-u_w>^5 + \frac{m_0}{6EI}<x-x_m>^3 - \frac{m_0}{6EI}<x-u_m>^3 \tag{2}$$

$$\theta_b = \theta_a + \frac{V_a L^2}{2EI} + \frac{M_a L}{EI} - \frac{P}{2EI}(L-x_P)^2 + \frac{K}{EI}(L-x_K) - \frac{w_0}{6EI}(L-x_w)^3 - \frac{w_1-w_0}{24EI(u_w-x_w)}(L-x_w)^4 + \frac{w_1}{6EI}(L-u_w)^3 + \frac{w_1-w_0}{24EI(u_w-x_w)}(L-u_w)^4 + \frac{m_0}{2EI}(L-x_m)^2 - \frac{m_0}{2EI}(L-u_m)^2 \tag{3}$$

$$y_b = y_a + \theta_a L + \frac{V_a L^3}{6EI} + \frac{M_a L^2}{2EI} - \frac{P}{6EI}(L-x_P)^3 + \frac{K}{2EI}(L-x_K)^2 - \frac{w_0}{24EI}(L-x_w)^4 - \frac{w_1-w_0}{120EI(u_w-x_w)}(L-x_w)^5 + \frac{w_1}{24EI}(L-u_w)^4 + \frac{w_1-w_0}{120EI(u_w-x_w)}(L-u_w)^5 + \frac{m_0}{6EI}(L-x_m)^3 - \frac{m_0}{6EI}(L-u_m)^3 \tag{4}$$

Fig. 1. Loading, deflections, and formulas in the *Method of Model Formulas* for beams

■ Summary of rudiments of singularity functions:

Notice that the argument of a singularity function is enclosed by angle brackets (i.e., < >). The argument of a regular function continues to be enclosed by parentheses [i.e., ( )]. The rudiments of singularity functions include the following:[8,9]

$$< x-a >^n = (x-a)^n \quad \text{if} \quad x-a \geq 0 \quad \text{and} \quad n > 0 \tag{5}$$

$$< x-a >^n = 1 \quad \text{if} \quad x-a \geq 0 \quad \text{and} \quad n = 0 \tag{6}$$

$$< x-a >^n = 0 \quad \text{if} \quad x-a < 0 \quad \text{or} \quad n < 0 \tag{7}$$

$$\int_{-\infty}^{x} < x-a >^n dx = \frac{1}{n+1} < x-a >^{n+1} \quad \text{if} \quad n > 0 \tag{8}$$

$$\int_{-\infty}^{x} < x-a >^n dx = < x-a >^{n+1} \quad \text{if} \quad n \leq 0 \tag{9}$$

$$\frac{d}{dx} < x-a >^n = n < x-a >^{n-1} \quad \text{if} \quad n > 0 \tag{10}$$

$$\frac{d}{dx} < x-a >^n = < x-a >^{n-1} \quad \text{if} \quad n \leq 0 \tag{11}$$

Equations (6) and (7) imply that, in using singularity functions for beams, we take

$$b^0 = 1 \quad \text{for} \quad b \geq 0 \tag{12}$$

$$b^0 = 0 \quad \text{for} \quad b < 0 \tag{13}$$

■ Summary of sign conventions for beams:

In the *method of model formulas*, the adopted sign conventions for various model loadings on the beam and for deflections of the beam with a constant flexural rigidity *EI* are illustrated in Fig. 1. Notice the following key points:

- A *shear force* is *positive* if it acts upward on the left (or downward on the right) face of the beam element [e.g., $\mathbf{V}_a$ at the left end *a*, and $\mathbf{V}_b$ at the right end *b* in Fig. 1*(a)*].
- At ends of the beam, a *moment* is *positive* if it tends to cause compression in the top fiber of the beam [e.g., $\mathbf{M}_a$ at the left end *a*, and $\mathbf{M}_b$ at the right end *b* in Fig. 1*(a)*].
- If not at ends of the beam, a *moment* is *positive* if it tends to cause compression in the top fiber of the beam just to the right of the position where it acts [e.g., the concentrated moment $\mathbf{K} = K \circlearrowright$ and the uniformly distributed moment with intensity m_0 in Fig. 1*(a)*].
- A *concentrated force* or a *distributed force* applied to the beam is *positive* if it is directed downward [e.g., the concentrated force $\mathbf{P} = P \downarrow$, the linearly distributed force with intensity w_0 on the left side and intensity w_1 on the right side in Fig. 1*(a)*, where the distribution becomes uniform if $w_0 = w_1$].

The slopes and deflections of a beam displaced from *AB* to *ab* are shown in Fig. 1*(b)*. Note that

- A *positive slope* is a counterclockwise angular displacement [e.g., θ_a and θ_b in Fig. 1*(b)*].
- A *positive deflection* is an upward linear displacement [e.g., y_a and y_b in Fig. 1*(b)*].

II. Enriching Students' Study with Method of Model Formulas via Contrast in Solutions

Equations (1) through (4) are related to the beam and loading shown in Fig. 1; they are the *model formulas* in the new method. Their derivation (*not* a main concern in this paper) can be found in the paper that propounded the **method of model formulas.**[1] Note that L in the model formulas in Eqs. (1) through (4) is a *parameter* representing the *total length* of the beam. In other words, L is to be replaced by the *total length* of the beam segment, to which the model formulas are applied. Statically indeterminate reactions as well as slopes and deflections of beams can, of course, be solved. A beam needs to be divided into multiple segments for analysis only if *(a)* it is a combined beam (e.g., a *Gerber beam*) having discontinuities in slope at hinge connections between segments, and *(b)* it contains segments with different flexural rigidities (e.g., a stepped beam). Having learned an additional efficacious method, students' study and set of skills are enriched.

Mechanics is mostly a deductive science, but learning is mostly an inductive process. For the purposes of **teaching** and **learning**, all examples will be **first** solved by the traditional *method of integration* (**MoI**) — *with* the use of singularity functions — **then** solved again by the *method of model formulas* (**MoMF**). As usual, the loading function, shear force, bending moment, slope, and deflection of the beam are denoted by the symbols q, V, M, y', and y, respectively.

Example 1. A cantilever beam AB with constant flexural rigidity EI and length L is acted on by a concentrated force of magnitude P at C, and two concentrated moments of magnitudes PL and $2PL$ at A and D, respectively, as shown in Fig. 2. Determine the slope θ_A and deflection y_A at end A.

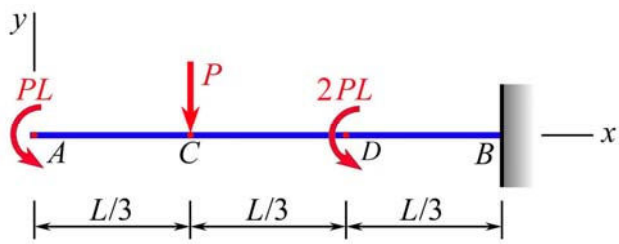


Fig. 2. Cantilever beam carrying a force and two moments

Solution by MoI. Using the symbols defined earlier and applying the *method of integration* (*with* the use of singularity functions) to this beam, we write

$$q = -PL < x >^{-2} - P < x - \frac{L}{3} >^{-1} - 2PL < x - \frac{2L}{3} >^{-2}$$

$$V = -PL < x >^{-1} - P < x - \frac{L}{3} >^{0} - 2PL < x - \frac{2L}{3} >^{-1}$$

$$EIy'' = M = -PL < x >^{0} - P < x - \frac{L}{3} >^{1} - 2PL < x - \frac{2L}{3} >^{0}$$

$$EIy' = -PL < x >^{1} - \frac{P}{2} < x - \frac{L}{3} >^{2} - 2PL < x - \frac{2L}{3} >^{1} + C_1$$

$$EIy = -\frac{PL}{2} < x >^{2} - \frac{P}{6} < x - \frac{L}{3} >^{3} - PL < x - \frac{2L}{3} >^{2} + C_1 x + C_2$$

The **boundary conditions** are $y'(L)=0$ and $y(L)=0$ at the fixed end *B*. Imposing these *two* conditions, respectively, we write

$$0=-PL(L)-\frac{P}{2}\left(\frac{2L}{3}\right)^2-2PL\left(\frac{L}{3}\right)+C_1$$

$$0=-\frac{PL}{2}(L)^2-\frac{P}{6}\left(\frac{2L}{3}\right)^3-PL\left(\frac{L}{3}\right)^2+C_1L+C_2$$

These *two* simultaneous equations yield

$$C_1=\frac{17PL^2}{9} \qquad\qquad C_2=-\frac{199PL^3}{162}$$

Substituting the values of C_1 and C_2 into the foregoing equations for EIy' and EIy, we write

$$\theta_A=y'\big|_{x=0}=\frac{C_1}{EI}=\frac{17PL^2}{9EI} \qquad\qquad y_A=y\big|_{x=0}=\frac{C_2}{EI}=-\frac{199PL^3}{162EI}$$

We report that

$$\theta_A=\frac{17\,PL^2}{9\,EI}\ \circlearrowleft \qquad\qquad y_A=\frac{199\,PL^3}{162\,EI}\ \downarrow$$

<u>Solution by MoMF</u>. In applying the *method of model formulas* to this beam, we make sure to adhere to the sign conventions as illustrated in Fig. 1. At end *A*, the moment M_A is $-PL$ and the shear force V_A is zero. At end *B*, the slope θ_B and deflection y_B are both zero. Note in the model formulas that we have $x_P=L/3$ for the concentrated force at *C*; $K=-2PL$ and $x_K=2L/3$ for the concentrated moment at *D*. Applying the model formulas in Eqs. (3) and (4), successively, to this beam *AB*, we write

$$0=\theta_A+0+\frac{-PL(L)}{EI}-\frac{P}{2EI}\left(L-\frac{L}{3}\right)^2+\frac{-2PL}{EI}\left(L-\frac{2L}{3}\right)-0-0+0+0+0-0$$

$$0=y_A+\theta_A L+0+\frac{-PL(L^2)}{2EI}-\frac{P}{6EI}\left(L-\frac{L}{3}\right)^3+\frac{-2PL}{2EI}\left(L-\frac{2L}{3}\right)^2-0-0+0+0+0-0$$

These *two* simultaneous equations yield

$$\theta_A=\frac{17\,PL^2}{9\,EI} \qquad\qquad y_A=-\frac{199\,PL^3}{162\,EI}$$

We report that

$$\theta_A=\frac{17\,PL^2}{9\,EI}\ \circlearrowleft \qquad\qquad y_A=\frac{199\,PL^3}{162\,EI}\ \downarrow$$

<u>Remark</u>. We observe that both the *method of integration* (*with* the use of singularity functions) and the *method of model formulas* yield the same solutions, as expected. In fact, the solution by the **MoMF** looks more direct than that by the **MoI**. Furthermore, if singularity functions were *not* used in the **MoI**, the solution would require division of the beam into multiple segments (such as *AC*, *CD*, and *DB*), and much more algebraic work in the solution would be involved. In Examples 2 through 4, readers may observe similar features.

Example 2. A beam AB with constant flexural rigidity EI and length L, a roller support at A, a fixed support at B, and carrying a distributed load of intensity w is shown in Fig. 3. Determine *(a)* the vertical reaction force $\mathbf{A}_y$ at A, *(b)* the slope θ_A at A, *(c)* the deflection y_C at C.

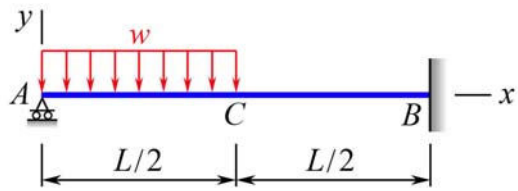


Fig. 3. Propped cantilever beam carrying a uniformly distributed load

Solution by MoI. We note that this beam AB is statically indeterminate to the *first* degree, and we may assume that $\mathbf{A}_y$ acts upward at A as shown in Fig. 4.

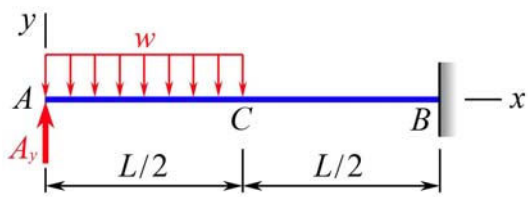


Fig. 4. Vertical reaction force $\mathbf{A}_y$ at A of the propped cantilever beam

Applying the *method of integration* to this beam, we write

$$q = A_y < x >^{-1} - w < x >^0 + w < x - \frac{L}{2} >^0$$

$$V = A_y < x >^0 - w < x >^1 + w < x - \frac{L}{2} >^1$$

$$EIy'' = M = A_y < x >^1 - \frac{w}{2} < x >^2 + \frac{w}{2} < x - \frac{L}{2} >^2$$

$$EIy' = \frac{A_y}{2} < x >^2 - \frac{w}{6} < x >^3 + \frac{w}{6} < x - \frac{L}{2} >^3 + C_1$$

$$EIy = \frac{A_y}{6} < x >^3 - \frac{w}{24} < x >^4 + \frac{w}{24} < x - \frac{L}{2} >^4 + C_1 x + C_2$$

The **boundary conditions** are $y'(L) = 0$ and $y(L) = 0$ at the fixed end B, as well as $y(0) = 0$ at the roller support A. Imposing these *three* conditions, respectively, we write

$$0 = \frac{A_y}{2} L^2 - \frac{w}{6} L^3 + \frac{w}{6}\left(\frac{L}{2}\right)^3 + C_1$$

$$0 = \frac{A_y}{6} L^3 - \frac{w}{24} L^4 + \frac{w}{24}\left(\frac{L}{2}\right)^4 + C_1 L + C_2$$

$$0 = C_2$$

These *three* simultaneous equations yield

$$C_1 = -\frac{11wL^3}{768} \qquad C_2 = 0 \qquad A_y = \frac{41wL}{128}$$

Substituting the values of C_1, C_2, and A_y into the foregoing equations for EIy' and EIy, we write

$$\theta_A = y'\big|_{x=0} = \frac{C_1}{EI} = -\frac{11wL^3}{768\,EI}$$

$$y_C = y\big|_{x=L/2} = \frac{1}{EI}\left[\frac{A_y}{6}(L/2)^3 - \frac{w}{24}(L/2)^4 + 0 + C_1(L/2) + 0\right] = -\frac{19wL^4}{6144\,EI}$$

We report that

$$\mathbf{A}_y = \frac{41wL}{128}\uparrow \qquad \theta_A = \frac{11wL^3}{768EI}\circlearrowright \qquad y_C = \frac{19wL^4}{6144EI}\downarrow$$

<u>Solution by MoMF</u>. Let the *method of model formulas* be now applied to solve for the statically indeterminate reaction $\mathbf{A}_y$ and the deflections of the beam. Upon inspecting the boundary conditions of this beam, we see that the deflection y_A at A, the moment M_A at A, the deflection y_B at B, and the slope θ_B at B are all equal to zero. The shear force at A is A_y. Noting that $x_w = 0$, $u_w = L/2$, and $w_0 = w_1 = w$, we apply the model formulas in Eqs. (3) and (4), successively, to the entire beam to write

$$0 = \theta_A + \frac{A_y L^2}{2EI} + 0 - 0 + 0 - \frac{w}{6EI}L^3 - 0 + \frac{w}{6EI}\left(L - \frac{L}{2}\right)^3 + 0 + 0 - 0$$

$$0 = 0 + \theta_A L + \frac{A_y L^3}{6EI} + 0 - 0 + 0 - \frac{w}{24EI}L^4 - 0 + \frac{w}{24EI}\left(L - \frac{L}{2}\right)^4 + 0 + 0 - 0$$

These *two* simultaneous equations yield

$$A_y = \frac{41wL}{128} \qquad \theta_A = -\frac{11wL^3}{768EI}$$

We report that

$$\mathbf{A}_y = \frac{41wL}{128}\uparrow \qquad \theta_A = \frac{11wL^3}{768EI}\circlearrowright$$

Using the above values of A_y and θ_A and letting $x = L/2$ in the model formula in Eq. (2), we write

$$y_C = y\big|_{x=L/2} = 0 + \theta_A\left(\frac{L}{2}\right) + \frac{A_y}{6EI}\left(\frac{L}{2}\right)^3 + 0 - 0 + 0 - \frac{w}{24EI}\left(\frac{L}{2}\right)^4 - 0 + 0 + 0 + 0 - 0$$

$$= -\frac{19wL^4}{6144EI}$$

We report that

$$y_C = \frac{19wL^4}{6144EI}\downarrow$$

Example 3. A cantilever beam AB with constant flexural rigidity EI and total length of $2L$ is propped at its midpoint C and carries a concentrated moment $\mathbf{M}_0$ as well as a distributed load of intensity w as shown in Fig. 5. Determine *(a)* the vertical reaction force $\mathbf{C}_y$ at C, *(b)* the slope θ_A at A, *(c)* the deflection y_A at A, *(d)* the slope θ_C at C.

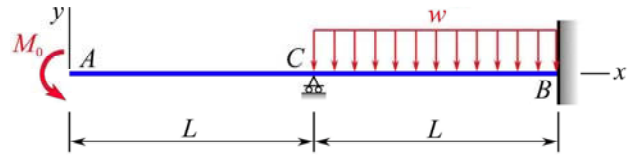


Fig. 5. Cantilever beam propped at its midpoint and carrying loads

Solution by MoI. We note that this beam AB is statically indeterminate to the *first* degree, and we may assume that $\mathbf{C}_y$ acts upward at C as shown in Fig. 6.

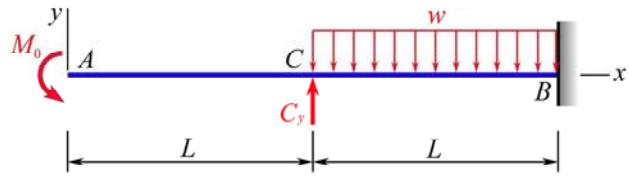


Fig. 6. Vertical reaction force $\mathbf{C}_y$ at C of the propped cantilever beam

Applying the *method of integration* to this beam, we write

$$q = -M_0 < x >^{-2} + C_y < x - L >^{-1} - w < x - L >^0$$

$$V = -M_0 < x >^{-1} + C_y < x - L >^0 - w < x - L >^1$$

$$EIy'' = M = -M_0 < x >^0 + C_y < x - L >^1 - \frac{w}{2} < x - L >^2$$

$$EIy' = -M_0 < x >^1 + \frac{C_y}{2} < x - L >^2 - \frac{w}{6} < x - L >^3 + C_1$$

$$EIy = -\frac{M_0}{2} < x >^2 + \frac{C_y}{6} < x - L >^3 - \frac{w}{24} < x - L >^4 + C_1 x + C_2$$

The **boundary conditions** are $y'(2L) = 0$ and $y(2L) = 0$ at the fixed end B, as well as $y(L) = 0$ at the roller support C. Imposing these *three* conditions, respectively, we write

$$0 = -M_0(2L) + \frac{C_y}{2} L^2 - \frac{w}{6} L^3 + C_1$$

$$0 = -\frac{M_0}{2}(2L)^2 + \frac{C_y}{6} L^3 - \frac{w}{24} L^4 + C_1(2L) + C_2$$

$$0 = -\frac{M_0}{2} L^2 + C_1 L + C_2$$

These *three* simultaneous equations yield

$$C_1 = \frac{L}{48}(60M_0 - wL^2) \qquad C_2 = -\frac{L^2}{48}(36M_0 - wL^2) \qquad C_y = \frac{3}{8L}(4M_0 + wL^2)$$

Substituting the values of C_1, C_2, and C_y into the foregoing equations for EIy' and EIy, we write

$$\theta_A = y'\big|_{x=0} = \frac{C_1}{EI} = \frac{L}{48EI}(60M_0 - wL^2)$$

$$y_A = y\big|_{x=0} = \frac{C_2}{EI} = -\frac{L^2}{48EI}(36M_0 - wL^2)$$

$$\theta_C = y'\big|_{x=L} = \frac{1}{EI}(-M_0L + C_1) = \frac{L}{48EI}(12M_0 - wL^2)$$

We report that

$$\mathbf{C}_y = \frac{3}{8L}(4M_0 + wL^2)\uparrow \qquad \theta_A = \frac{L}{48\,EI}(60M_0 - wL^2)\circlearrowleft$$

$$y_A = \frac{L^2}{48\,EI}(36M_0 - wL^2)\downarrow \qquad \theta_C = \frac{L}{48\,EI}(12M_0 - wL^2)\circlearrowleft$$

<u>Solution by MoMF</u>. Let the *method of model formulas* be now applied to solve the problem. The reaction force $\mathbf{C}_y$ at C in Fig. 6 may be treated as an unknown applied concentrated force. We note that this beam has a total length of $2L$, which will be the value for the *parameter* L in the model formulas in Eqs. (1) through (4). Upon inspecting the **boundary conditions** of this beam, we see that the deflection y_B and the slope θ_B at the fixed end B, as well as the deflection y_C at C, are equal to zero. Applying Eqs. (3) and (4) to the beam AB and using Eq. (2) to impose the condition that $y_C = 0$ at C, in that *order*, we write

$$0 = \theta_A + 0 + \frac{-M_0(2L)}{EI} - \frac{-C_y}{2EI}(2L-L)^2 + 0 - \frac{w}{6EI}(2L-L)^3 - 0 + 0 + 0 + 0 - 0$$

$$0 = y_A + \theta_A(2L) + 0 + \frac{M_0(2L)^2}{2EI} - \frac{-C_y}{6EI}(2L-L)^3 + 0 - \frac{w}{24EI}(2L-L)^4 - 0 + 0 + 0 + 0 - 0$$

$$0 = y_A + \theta_A L + 0 + \frac{-M_0}{2EI}L^2 - 0 + 0 - 0 - 0 + 0 + 0 + 0 - 0$$

These *three* simultaneous equations yield

$$C_y = \frac{3}{8L}(4M_0 + wL^2) \qquad \theta_A = \frac{L}{48EI}(60M_0 - wL^2) \qquad y_A = -\frac{L^2}{48EI}(36M_0 - wL^2)$$

Using the above value for θ_A and letting $x = L$ in Eq. (1), we write

$$\theta_C = y'\big|_{x=L} = \theta_A + 0 + \frac{-M_0}{EI}L - 0 + 0 - 0 - 0 + 0 + 0 + 0 - 0 = \frac{L}{48EI}(12M_0 - wL^2)$$

We report that

$$\mathbf{C}_y = \frac{3}{8L}(4M_0 + wL^2)\uparrow \qquad \theta_A = \frac{L}{48\,EI}(60M_0 - wL^2)\circlearrowleft$$

$$y_A = \frac{L^2}{48\,EI}(36M_0 - wL^2)\downarrow \qquad \theta_C = \frac{L}{48\,EI}(12M_0 - wL^2)\circlearrowleft$$

Example 4. A continuous beam AB with constant flexural rigidity EI and total length $2L$ has a roller support at A, a roller support at C, a fixed support at B and carries a linearly distributed load as shown in Fig. 7. Determine *(a)* the vertical reaction force $\mathbf{A}_y$ and the slope θ_A at A, *(b)* the vertical reaction force $\mathbf{C}_y$ and the slope θ_C at C.

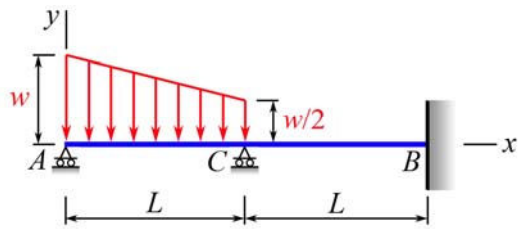


Fig. 7. Continuous beam carrying linearly distributed load

Solution by MoI. We note that the beam AB is statically indeterminate to the *second* degree, and we may assume that $\mathbf{A}_y$ and $\mathbf{C}_y$ act upward at A and C, respectively, as shown in Fig. 8.

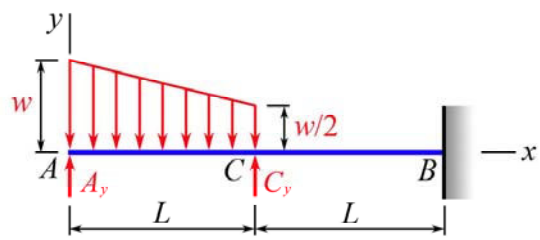


Fig. 8. Reaction forces $\mathbf{A}_y$ at A and $\mathbf{C}_y$ at C of the continuous beam

In applying the *method of integration* to this beam, we first use the concept of *superposition* to write the loading function q as follows:

$$q = A_y < x >^{-1} + C_y < x - L >^{-1} - w < x >^0 + \frac{w}{2L} < x >^1 - \frac{w}{2L} < x - L >^1 + \frac{w}{2} < x - L >^0$$

Then, we write

$$V = A_y < x >^0 + C_y < x - L >^0 - w < x >^1 + \frac{w}{4L} < x >^2 - \frac{w}{4L} < x - L >^2 + \frac{w}{2} < x - L >^1$$

$$EIy'' = M = A_y < x >^1 + C_y < x - L >^1 - \frac{w}{2} < x >^2 + \frac{w}{12L} < x >^3 - \frac{w}{12L} < x - L >^3 + \frac{w}{4} < x - L >^2$$

$$EIy' = \frac{A_y}{2} < x >^2 + \frac{C_y}{2} < x - L >^2 - \frac{w}{6} < x >^3 + \frac{w}{48L} < x >^4 - \frac{w}{48L} < x - L >^4 + \frac{w}{12} < x - L >^3 + C_1$$

$$EIy = \frac{A_y}{6} < x >^3 + \frac{C_y}{6} < x - L >^3 - \frac{w}{24} < x >^4 + \frac{w}{240L} < x >^5 - \frac{w}{240L} < x - L >^5 + \frac{w}{48} < x - L >^4 + C_1 x + C_2$$

The **boundary conditions** are $y'(2L) = 0$ and $y(2L) = 0$ at the fixed end B, $y(L) = 0$ at the roller support C, and $y(0) = 0$ at the roller support A. Imposing these *four* conditions, in *order*, we write

$$0 = \frac{A_y}{2}(2L)^2 + \frac{C_y}{2}L^2 - \frac{w}{6}(2L)^3 + \frac{w}{48L}(2L)^4 - \frac{w}{48L}L^4 + \frac{w}{12}L^3 + C_1$$

$$0 = \frac{A_y}{6}(2L)^3 + \frac{C_y}{6}L^3 - \frac{w}{24}(2L)^4 + \frac{w}{240L}(2L)^5 - \frac{w}{240L}L^5 + \frac{w}{48}L^4 + C_1(2L) + C_2$$

$$0 = \frac{A_y}{6}L^3 - \frac{w}{24}L^4 + \frac{w}{240L}L^5 + C_1L + C_2$$

$$0 = C_2$$

The above *four* simultaneous equations yield

$$C_1 = -\frac{13wL^3}{560} \qquad C_2 = 0 \qquad A_y = \frac{51wL}{140} \qquad C_y = \frac{13wL}{28}$$

Substituting the values of C_1 and A_y into the foregoing equation for EIy', we write

$$\theta_C = y'\big|_{x=L} = \frac{1}{EI}\left(\frac{A_y}{2}L^2 - \frac{w}{6}L^3 + \frac{w}{48L}L^4 + C_1\right) = \frac{11wL^3}{840EI}$$

We report that

$$\mathbf{A}_y = \frac{51wL}{140}\uparrow \qquad \theta_A = \frac{13wL^3}{560EI}\circlearrowright \qquad \mathbf{C}_y = \frac{13wL}{28}\uparrow \qquad \theta_C = \frac{11wL^3}{840EI}\circlearrowleft$$

<u>Solution by MoMF</u>. Let the *method of model formulas* be now applied to solve the problem. The reaction force $\mathbf{C}_y$ at C in Fig. 8 may be treated as an unknown applied concentrated force. We notice that the beam AB has a total length of $2L$, which will be the value for the *parameter* L in the model formulas in Eqs. (1) through (4). Upon inspecting the **boundary conditions** of this beam, we see that the moment M_A and deflection y_A at A are zero, the slope θ_B and deflection y_B at B are zero, and the deflection y_C at C is zero. The shear force at the left end A is the vertical reaction force $\mathbf{A}_y$ at A. Applying Eqs. (3) and (4) to the beam AB and using Eq. (2) to impose the condition that $y_C = 0$ at C, in that *order*, we write

$$0 = \theta_A + \frac{A_y(2L)^2}{2EI} + 0 - \frac{-C_y}{2EI}(2L-L)^2 + 0 - \frac{w}{6EI}(2L)^3 - \frac{(w/2)-w}{24EIL}(2L)^4$$
$$+ \frac{w/2}{6EI}(2L-L)^3 + \frac{(w/2)-w}{24EIL}(2L-L)^4 + 0 - 0$$

$$0 = 0 + \theta_A(2L) + \frac{A_y(2L)^3}{6EI} + 0 - \frac{-C_y}{6EI}(2L-L)^3 + 0 - \frac{w}{24EI}(2L)^4 - \frac{(w/2)-w}{120EIL}(2L)^5$$
$$+ \frac{w/2}{24EI}(2L-L)^4 + \frac{(w/2)-w}{120EIL}(2L-L)^5 + 0 - 0$$

$$0 = 0 + \theta_A L + \frac{A_y}{6EI}L^3 + 0 - 0 + 0 - \frac{w}{24EI}L^4 - \frac{(w/2)-w}{120EIL}L^5 + 0 + 0 + 0 - 0$$

These *three* simultaneous equations yield

$$A_y = \frac{51wL}{140} \qquad \theta_A = -\frac{13wL^3}{560EI} \qquad C_y = \frac{13wL}{28}$$

We report that

$$\mathbf{A}_y = \frac{51wL}{140}\uparrow \qquad \theta_A = \frac{13wL^3}{560EI}\circlearrowright \qquad \mathbf{C}_y = \frac{13wL}{28}\uparrow$$

The slope θ_C is simply y' evaluated at C, which is located at $x = L$. Applying Eq. (1) and using the above values for θ_A and A_y, we write

$$\theta_C = y'\big|_{x=L} = \theta_A + \frac{A_y}{2EI}L^2 + 0 - 0 + 0 - \frac{w}{6EI}L^3 - \frac{(w/2)-w}{24EIL}L^4 + 0 + 0 + 0 - 0 = \frac{11wL^3}{840EI}$$

We report that

$$\theta_C = \frac{11wL^3}{840EI}\circlearrowleft$$

III. Assessment and an Effective Approach to Teaching the MoMF

The *method of model formulas* is a general methodology that employs a set of four equations to serve as model formulas in solving problems involving statically indeterminate reactions, as well as the slopes and deflections, of elastic beams. The first two model formulas are for the slope and deflection at any position x of the beam and contain rudimentary singularity functions, while the other two model formulas contain only traditional algebraic expressions. Generally, this method requires much less effort in solving beam deflection problems. Most students favor this method because they can solve problems in shorter time using this method and they score higher in tests.

The examples in Section II provide a variety of head-to-head comparisons between solutions by the traditional *method of integration* and those by the *method of model formulas*; and all of the solutions are, respectively, in agreement. Thus, all solutions by the *method of model formulas* are naturally correct. The writer has been successful in effectively introducing and teaching the *method of model formulas* to students to enrich their study and set of skills in determining statically indeterminate reactions and deflections of elastic beams by using the following steps:

- Teach the traditional *method of integration* and the imposition of boundary conditions.
- Teach the rudiments of *singularity functions* and utilize them in the *method of integration*.
- Go over briefly the derivation[1] of the *four model formulas* in terms of *singularity functions*.
- Give students the heads-up on the following advantages in the *method of model formulas*:
  - No need of integration or evaluation of constants of integration.
  - Not prone to generate a large number of simultaneous equations even if
    - ▷ the beam carries multiple concentrated loads (forces or moments),
    - ▷ the beam has one or more simple supports *not* at its ends,
    - ▷ the beam has linearly distributed loads *not* starting at its left end, and
    - ▷ the beam has linearly distributed loads *not* ending at its right end.
- Demonstrate solutions of several beam problems by the *method of model formulas*.
- Assess the solutions obtained (e.g., comparing with solutions by another method).

Although solutions obtained by the *method of model formulas* are often more direct than those obtained by the *method of integration*, a **one-page excerpt** from the *method of model formulas*,

such as that shown in Fig. 1, must be made available to those who used this method. Still, one may recall that a **table of formulas** for slope and deflection of selected beams having a variety of supports and loading is *also* needed by persons who use the method of superposition. In this regard, the *method of model formulas* is **on a par with** the *method of superposition*.

IV. Concluding Remarks

In the *method of model formulas*, no explicit integration or differentiation is involved in applying any of the model formulas. The model formulas essentially serve to provide *material equations* (which involve and reflect the material property) besides the equations of static equilibrium of the beam that can readily be written. Selected applied loads are illustrated in Fig. 1*(a)*, which cover most of the loads encountered in undergraduate Mechanics of Materials. In the case of a nonlinearly distributed load on the beam, the model formulas may be modified by the user for a specific nonlinearly distributed load.

The *method of model formulas* is best taught to students as an alternative method, after they have learned one or more of the traditional methods.[2-12] This method enriches students' study and set of skills in their determining reactions and deflections of beams, and it provides engineers with a means to quickly check their solutions obtained using traditional methods.

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