Teaching Deflections of Beams: Advantages of Method of Model Formulas versus Method of Integration

Ing-Chang Jong, William T. Springer, Rick J. Couvillion
University of Arkansas, Fayetteville, AR 72701

Abstract

The method of model formulas is a new method for solving statically indeterminate reactions and deflections of elastic beams. Since its publication in a recent issue of the IJEE, many instructors of Mechanics of Materials have considerable interest in knowing an effective way for teaching this method to enrich students’ study and their set of skills in determining beam reactions and deflections. Moreover, people are interested in seeing demonstrations showing any advantage of this method over the traditional methods. This paper is aimed at (a) providing comparisons of this new method versus the traditional method of integration via several head-to-head contrasting solutions of same problems, and (b) proposing a set of steps for use to effectively introduce and teach this new method to students. It is a considered opinion that the method of model formulas be taught to students after having taught them one or more of the traditional methods.

1. Introduction

Beams are longitudinal members subjected to transverse loads. Students usually first learn the design of beams for strength. Then they learn the determination of deflections of beams under a variety of loads. Traditional methods used in determining statically indeterminate reactions and deflections of elastic beams include: method of integration (with or without use of singularity functions), method of superposition, method using moment-area theorems, method of conjugate beam, method using Castigliano’s theorem, and method of segments.

The method of model formulas is a newly propounded method. Beginning with an elastic beam under a preset general loading, a set of four model formulas are derived and established for use in this new method. These formulas are expressed in terms of the following:

- (a) flexural rigidity of the beam;
- (b) slopes, deflections, shear forces, and bending moments at both ends of the beam;
- (c) typical applied loads (concentrated force, concentrated moment, linearly distributed force, and uniformly distributed moment) somewhere on the beam.

For starters, one must know that a working proficiency in the rudiments of singularity functions is a prerequisite to using the method of model formulas. To benefit a wider readership, which may have different specialties in mechanics, and to avoid or minimize any possible misunderstanding, this paper includes summaries of the rudiments of singularity functions and the sign conventions for beams. Readers, who are familiar with these topics, may skip the summaries. An excerpt from the method of model formulas is needed and shown in Fig. 1, courtesy of IJEE.
Excerpt from the Method of Model Formulas

Positive directions of forces, moments, slopes, and deflections

\[ y'(x) = \theta_w + \frac{V_w}{2EI} x^2 + \frac{M_w}{EI} x - \frac{P}{2EI} (x-x_p)^2 + \frac{K}{EI} (x-x_k)^3 - \frac{w}{6EI} (x-x_w)^4 \]

\[ y(x) = y'(x) + \theta_w x + \frac{V_w}{6EI} x^3 + \frac{M_w}{2EI} x^2 - \frac{P}{2EI} (x-x_p)^3 + \frac{K}{2EI} (x-x_k)^4 - \frac{w}{24EI} (x-x_w)^5 \]

\[ \theta_w = \theta_w + \frac{V_w L^2}{2EI} + \frac{M_w L}{EI} - \frac{P}{2EI} (L-x_p)^2 + \frac{K}{EI} (L-x_k)^3 - \frac{w}{6EI} (L-x_w)^4 \]

\[ y_0 = y_{w0} + \frac{V_w L}{6EI} + \frac{M_w L^2}{2EI} - \frac{P}{6EI} (L-x_p)^3 + \frac{K}{2EI} (L-x_k)^4 - \frac{w}{24EI} (L-x_w)^5 \]

Fig. 1. Loading, deflections, and formulas in the Method of Model Formulas for beams
Summary of rudiments of singularity functions:

Notice that the argument of a singularity function is enclosed by angle brackets (i.e., < >). The argument of a regular function continues to be enclosed by parentheses [i.e., ( )]. The rudiments of singularity functions include the following:\textsuperscript{8,9}

\[
<x-a>^n = (x-a)^n \text{ if } x-a \geq 0 \text{ and } n > 0 \tag{5}
\]

\[
<x-a>^n = 1 \text{ if } x-a \geq 0 \text{ and } n = 0 \tag{6}
\]

\[
<x-a>^n = 0 \text{ if } x-a < 0 \text{ or } n < 0 \tag{7}
\]

\[
\int_{-\infty}^{x} <x-a>^n \, dx = \frac{1}{n+1} <x-a>^{n+1} \text{ if } n > 0 \tag{8}
\]

\[
\int_{-\infty}^{x} <x-a>^n \, dx = <x-a>^{n+1} \text{ if } n \leq 0 \tag{9}
\]

\[
\frac{d}{dx} <x-a>^n = n <x-a>^{n-1} \text{ if } n > 0 \tag{10}
\]

\[
\frac{d}{dx} <x-a>^n = <x-a>^{n-1} \text{ if } n \leq 0 \tag{11}
\]

Equations (6) and (7) imply that, in using singularity functions for beams, we take

\[
b^0 = 1 \text{ for } b \geq 0 \tag{12}
\]

\[
b^0 = 0 \text{ for } b < 0 \tag{13}
\]

Summary of sign conventions for beams:

In the method of model formulas, the adopted sign conventions for various model loadings on the beam and for deflections of the beam with a constant flexural rigidity $EI$ are illustrated in Fig. 1. Notice the following key points:

- A shear force is positive if it acts upward on the left (or downward on the right) face of the beam element [e.g., $V_a$ at the left end $a$, and $V_b$ at the right end $b$ in Fig. 1(a)].

- At ends of the beam, a moment is positive if it tends to cause compression in the top fiber of the beam [e.g., $M_a$ at the left end $a$, and $M_b$ at the right end $b$ in Fig. 1(a)].

- If not at ends of the beam, a moment is positive if it tends to cause compression in the top fiber of the beam just to the right of the position where it acts [e.g., the concentrated moment $K = K_0 \bigcup$ and the uniformly distributed moment with intensity $m_0$ in Fig. 1(a)].

- A concentrated force or a distributed force applied to the beam is positive if it is directed downward [e.g., the concentrated force $P = P \downarrow$, the linearly distributed force with intensity $u_0$ on the left side and intensity $u_1$ on the right side in Fig. 1(a), where the distribution becomes uniform if $w_0 = w_1$].

The slopes and deflections of a beam displaced from $AB$ to $ab$ are shown in Fig. 1(b). Note that

- A positive slope is a counterclockwise angular displacement [e.g., $\theta_a$ and $\theta_b$ in Fig. 1(b)].

- A positive deflection is an upward linear displacement [e.g., $y_a$ and $y_b$ in Fig. 1(b)].
II. Teaching and Learning a New Method via Contrast between Solutions

Equations (1) through (4) are related to the beam and loading shown in Fig. 1; they are the *model formulas* in the new method. Their derivation (not a main concern in this paper) can be found in the paper that propounded the *method of model formulas*. Note that $L$ in the model formulas in Eqs. (1) through (4) is a *parameter* representing the *total length* of the beam. In other words, $L$ is to be replaced by the *total length* of the beam segment, to which the model formulas are applied. Statically indeterminate reactions as well as slopes and deflections of beams can, of course, be solved. A beam needs to be divided into segments for analysis only if (a) it is a combined beam (e.g., a *Gerber beam*) having discontinuities in slope at hinge connections between segments, and (b) it contains segments with different flexural rigidities (e.g., a stepped beam). Having learned an additional efficacious method, students’ study and set of skills are enriched.

Mechanics is mostly a deductive science, but learning is mostly an inductive process. For the purposes of teaching and learning, all examples will be *first* solved by the traditional *method of integration* (MoI) — with use of singularity functions — then solved again by the *method of model formulas* (MoMF). As usual, the loading function, shear force, bending moment, slope, and deflection of the beam are denoted by the symbols $q$, $V$, $M$, $y'$, and $y$, respectively.

**Example 1.** A simply supported beam $AD$ with constant flexural rigidity $EI$ and length $L$ is acted on by a concentrated force $P \downarrow$ at $B$ and a concentrated moment $PL \uparrow$ at $C$ as shown in Fig. 2. Determine (a) the slopes $\theta_A$ and $\theta_D$ at $A$ and $D$, respectively; (b) the deflection $y_B$ at $B$.

![Fig. 2. Simply supported beam AD carrying concentrated loads](image)

**Solution.** The beam is in static equilibrium. Its free-body diagram is shown in Fig. 3.

![Fig. 3. Free-body diagram of the simply supported beam AD](image)

- **Using MoI:** Using the symbols defined earlier and applying the *method of integration* (with use of singularity functions) to this beam, we write

$$ q = \frac{5P}{3} < x >^{-1} - P < x - \frac{L}{3} >^{-1} - PL < x - \frac{2L}{3} >^{-2} $$

$$ V = \frac{5P}{3} < x >^{0} - P < x - \frac{L}{3} >^{0} - PL < x - \frac{2L}{3} >^{0} $$

$$ M = EIy'' = \frac{5P}{3} < x >^{1} - P < x - \frac{L}{3} >^{1} - PL < x - \frac{2L}{3} >^{0} $$
\[ E I y'' = \frac{5P}{6} < x >^2 - \frac{P}{2} < x - \frac{L}{3} >^2 - PL < x - \frac{2L}{3} >^3 + C_1 \]
\[ E I y = \frac{5P}{18} < x >^3 - \frac{P}{6} < x - \frac{L}{3} >^3 - \frac{PL}{2} < x - \frac{2L}{3} >^3 + C_1 x + C_2 \]

The boundary conditions of this beam reveal that \( y(0) = 0 \) at \( A \) and \( y(L) = 0 \) at \( D \). Imposing these two conditions, respectively, we write
\[ 0 = C_2 \]
\[ 0 = \frac{5P}{18} (L)^3 - \frac{P}{6} \left( \frac{2L}{3} \right)^3 - \frac{PL}{2} \left( \frac{L}{3} \right)^2 + C_1 L \]

These two simultaneous equations yield
\[ C_1 = - \frac{14PL^2}{81} \quad C_2 = 0 \]

Using these values and the foregoing equations for \( E I y' \) and \( E I y \), we write
\[ \theta_A = y'' \big|_{x=0} = \frac{C_1}{EI} = - \frac{14PL^2}{81EI} \]
\[ \theta_D = y'' \big|_{x=L} = \frac{5P}{6EI} (L)^2 - \frac{P}{2EI} \left( \frac{2L}{3} \right)^3 - \frac{PL}{2EI} \left( \frac{L}{3} \right)^2 + \frac{C_1}{EI} = \frac{17PL^2}{162EI} \]
\[ y_B = y \big|_{x=L/3} = \frac{5P}{18EI} \left( \frac{L}{3} \right)^3 + \frac{C_1}{EI} \left( \frac{L}{3} \right) = - \frac{23PL^3}{486EI} \]

We report that
\[ \theta_A = \frac{14PL^2}{81EI} \quad \theta_D = \frac{17PL^2}{162EI} \quad y_B = \frac{23PL^3}{486EI} \]

**Using MoMF:** In applying the method of model formulas to this beam, we must adhere to the sign conventions as illustrated in Fig. 1. At the left end \( A \), the moment \( M_A \) is 0, the shear force \( V_A \) is \( 5P/3 \), the deflection \( y_A \) is 0, but the slope \( \theta_A \) is unknown. At the right end \( D \), the deflection \( y_D \) is 0, but the slope \( \theta_D \) is unknown. Note in the model formulas that we have \( x_p = L/3 \) for the concentrated force \( P \downarrow \) at \( B \) and \( x_p = 2L/3 \) for the concentrated moment \( PL \downarrow \) at \( C \). Applying the model formulas in Eqs. (3) and (4), successively, to this beam \( AD \), we write
\[ \theta_D = \theta_A + \frac{(5P/3)L^2}{2EI} + 0 - \frac{P}{2EI} \left( L - \frac{L}{3} \right)^2 + \frac{-PL}{EI} \left( L - \frac{2L}{3} \right) - 0 - 0 + 0 + 0 - 0 \]
\[ 0 = 0 + \theta_A L + \frac{(5P/3)L^3}{6EI} + 0 - \frac{P}{6EI} \left( L - \frac{L}{3} \right)^3 + \frac{-PL}{2EI} \left( L - \frac{2L}{3} \right)^2 - 0 - 0 + 0 + 0 - 0 \]

These two simultaneous equations yield
\[ \theta_A = - \frac{14PL^2}{81EI} \quad \theta_D = \frac{17PL^2}{162EI} \]

Using the value of \( \theta_A \) and applying the model formula in Eq. (2), we write
\[ y_B = y \big|_{x=L/3} = 0 + \theta_A \left( \frac{L}{3} \right)^3 + \frac{5P}{6EI} \left( \frac{L}{3} \right)^3 + 0 - 0 - 0 - 0 + 0 + 0 - 0 = \frac{23PL^3}{486EI} \]
We report that
\[ \theta_A = \frac{14PL^2}{81EI} \]
\[ \theta_D = \frac{17PL^2}{162EI} \]
\[ y_b = \frac{23PL^3}{486EI} \]

**Remark.** We observe that both the method of integration (with use of singularity functions) and the method of model formulas yield the same solutions, as expected. In fact, the solution by the MoMF looks more direct than that by the MoI. Furthermore, if singularity functions were not used in the MoI, the solution would require division of the beam into multiple segments (such as AB, BC, and CD), and much more effort in algebraic work in the solution would be involved. In Examples 2 through 5, readers may observe similar features.

**Example 2.** A cantilever beam AC with constant flexural rigidity EI and length L is loaded with a distributed load of intensity w in segment AB as shown in Fig. 4. Determine (a) the slope \( \theta_A \) and deflection \( y_A \) at A, (b) the slope \( \theta_B \) and deflection \( y_B \) at B.

![Diagram of cantilever beam AC](image)

**Solution.** The beam is in static equilibrium. Its free-body diagram is shown in Fig. 5.

![Free-body diagram of cantilever beam AC](image)

- **Using MoI:** Applying the method of integration to this beam, we write

  \[ q = -w < x >^0 + w < x - \frac{L}{2} >^0 \]
  \[ V = -w < x >^1 + w < x - \frac{L}{2} >^1 \]
  \[ M = Ely'' = -\frac{w}{2} < x >^2 + \frac{w}{2} < x - \frac{L}{2} >^2 \]
  \[ Ely' = -\frac{w}{6} < x >^3 + \frac{w}{6} < x - \frac{L}{2} >^3 + C_1 \]
  \[ Ely = -\frac{w}{24} < x >^4 + \frac{w}{24} < x - \frac{L}{2} >^4 + C_1 x + C_2 \]

The boundary conditions of this beam reveal that \( y'(L) = 0 \) and \( y(L) = 0 \) at C. Imposing these two conditions, respectively, we write

\[ 0 = -\frac{w}{6} L^3 + \frac{w}{6} \left( \frac{L}{2} \right)^3 + C_1 \]
These two simultaneous equations yield

\[ C_1 = \frac{7wL^3}{48} \quad C_2 = -\frac{41wL^4}{384} \]

Using these values and the foregoing equations for \( E\alpha \) and \( E\gamma \), we write

\[
\begin{align*}
\theta_A &= y'_A \big|_{x=0} = \frac{C_1}{EI} = \frac{7wL^3}{48EI} \\
y_A &= y_A \big|_{x=0} = \frac{C_2}{EI} = -\frac{41wL^4}{384EI} \\
\theta_B &= y'_B \big|_{x=L/2} = -\frac{w}{6EI} \left(\frac{L}{2}\right)^3 + \frac{C_1}{EI} = \frac{wL^3}{8EI} \\
y_B &= y_B \big|_{x=L/2} = -\frac{w}{24EI} \left(\frac{L}{2}\right)^4 + \frac{C_1}{EI} + \frac{C_2}{EI} = -\frac{7wL^4}{192EI}
\end{align*}
\]

We report that

\[
\begin{align*}
\theta_A &= \frac{7wL^3}{48EI} \\
y_A &= \frac{41wL^4}{384EI} \\
\theta_B &= \frac{wL^3}{8EI} \\
y_B &= \frac{7wL^4}{192EI}
\end{align*}
\]

- **Using MoMF:** Let the *method of model formulas* be now applied to this beam. The shear force\( V_A \) and bending moment \( M_A \) at the free end \( A \), as well as the slope \( \theta_c \) and deflection \( y_c \) at the fixed end \( C \), are all zero. Noting that \( x_w = 0 \) and \( u_w = L/2 \), we apply the model formulas in Eqs. (3) and (4) to the entire beam to write

\[
0 = \theta_A + 0 + 0 + 0 - \frac{w}{6EI} L^3 - 0 + \frac{w}{6EI} \left( L - \frac{L}{2} \right)^3 + 0 + 0 - 0
\]

\[
0 = y_A + \theta_A L + 0 + 0 + 0 - \frac{w}{24EI} L^4 - 0 + \frac{w}{24EI} \left( L - \frac{L}{2} \right)^4 + 0 + 0 - 0
\]

These two simultaneous equations yield

\[
\begin{align*}
\theta_A &= \frac{7wL^3}{48EI} \\
y_A &= -\frac{41wL^4}{384EI}
\end{align*}
\]

Using these values and applying the model formulas in Eqs. (1) and (2), respectively, we write

\[
\begin{align*}
\theta_B &= y'_B \big|_{x=L/2} = \theta_A + 0 + 0 - 0 - \frac{w}{6EI} \left( \frac{L}{2} \right)^3 - 0 + 0 + 0 - 0 = \frac{wL^3}{8EI} \\
y_B &= y_B \big|_{x=L/2} = y_A + \theta_A \left( \frac{L}{2} \right) + 0 + 0 - 0 - \frac{w}{24EI} \left( \frac{L}{2} \right)^4 - 0 + 0 + 0 + 0 - 0 = -\frac{7wL^4}{192EI}
\end{align*}
\]

We report that

\[
\begin{align*}
\theta_A &= \frac{7wL^3}{48EI} \\
y_A &= \frac{41wL^4}{384EI} \\
\theta_B &= \frac{wL^3}{8EI} \\
y_B &= \frac{7wL^4}{192EI}
\end{align*}
\]
Example 3. A cantilever beam $AC$ with constant flexural rigidity $EI$ and total length $2L$ is propped at $A$ and carries a concentrated moment $M_0 \triangle$ at $B$ as shown in Fig. 6. Determine (a) the vertical reaction force $A_y$ and slope $\theta_A$ at $A$, (b) the slope $\theta_B$ and deflection $y_B$ at $B$.

![Image](https://example.com/image.png)

**Fig. 6.** Cantilever beam $AC$ propped at $A$ and carrying a moment at $B$

**Solution.** The beam is in static equilibrium. Its free-body diagram is shown in Fig. 7, where we note that the beam is statically indeterminate to the first degree.

![Image](https://example.com/image.png)

**Fig. 7.** Free-body diagram of the propped cantilever beam $AC$

- **Using MoI:** Applying the *method of integration* to this beam, we write

  \begin{align*}
  q &= A_y < x >^{-1} - M_0 < x - L >^2 \\
  V &= A_y < x >^0 - M_0 < x - L >^1 \\
  M &= E I y'' = A_y < x >^3 - M_0 < x - L >^0 \\
  E I y' &= \frac{A_y}{2} < x >^2 - M_0 < x - L >^1 + C_1 \\
  E I y &= \frac{A_y}{6} < x >^3 - \frac{M_0}{2} < x - L >^2 + C_1 x + C_2
  \end{align*}

  The *boundary conditions* of this beam reveal that $y(0) = 0$ at $A$, $y'(2L) = 0$ at $C$, and $y(2L) = 0$ at $C$. Imposing these three conditions, respectively, we write

  \begin{align*}
  0 &= C_2 \\
  0 &= \frac{A_y}{2} (2L)^2 - M_0 L + C_1 \\
  0 &= \frac{A_y}{6} (2L)^3 - \frac{M_0}{2} L^2 + C_1 (2L) + C_2
  \end{align*}

These three simultaneous equations yield

\[ C_1 = - \frac{M_0 L}{8}, \quad C_2 = 0, \quad A_y = \frac{9M_0}{16L} \]

Using these values and the foregoing equations for $E I y'$ and $E I y$, we write

\begin{align*}
\theta_A &= y'|_{x=0} = \frac{C_1}{E I} = - \frac{M_0 L}{8 E I} \\
\theta_B &= y'|_{x=L} = \frac{A_y}{2 E I} L^2 + \frac{C_1}{E I} = \frac{5M_0 L}{32 E I} \\
y_B &= y|_{x=L} = \frac{A_y}{6 E I} L^3 + \frac{C_1}{E I} L = - \frac{M_0 L^2}{32 E I}
\end{align*}

We report that...
Using MoMF: Let the method of model formulas be now applied to this beam. We note that this beam has a total length of $2L$, which will be the value for the parameter $L$ in all the model formulas in Eqs. (1) through (4). We see that the deflection $y_C$ and the slope $\theta_C$ at $C$, as well as the deflection $y_A$ at $A$, are equal to zero. Applying the model formulas in Eqs. (3) and (4) to the entire beam, we write

$$0 = \theta_A + \frac{A_y(2L)^2}{2EI} + 0 - 0 + \frac{-M_0}{EI}(2L - L) - 0 - 0 + 0 + 0 - 0$$

$$0 = 0 + \theta_A(2L) + \frac{A_y(2L)^3}{6EI} + 0 - 0 + \frac{-M_0}{2EI}(2L - L)^2 - 0 - 0 + 0 + 0 + 0 - 0$$

These two simultaneous equations yield

$$A_y = \frac{9M_0}{16L} \quad \theta_A = -\frac{M_0L}{8EI}$$

Using these values and applying the model formulas in Eqs. (1) and (2), respectively, we write

$$\theta_B = y'_A|_{x=L} = \theta_A + \frac{A_y}{2EI}L^2 + 0 - 0 + 0 - 0 + 0 + 0 + 0 + 0 = \frac{5M_0L}{32EI}$$

$$y_B = y'_A|_{x=L} = 0 + \theta_A L + \frac{A_y}{6EI}L^3 + 0 - 0 + 0 - 0 + 0 + 0 + 0 + 0 - 0 = -\frac{M_0L^2}{32EI}$$

We report that

$$A_y = \frac{9M_0}{16L} \quad \theta_A = \frac{M_0L}{8EI} \quad \theta_B = \frac{5M_0L}{32EI} \quad y_B = \frac{M_0L^2}{32EI}$$

**Example 4.** A continuous beam $AC$ with constant flexural rigidity $EI$ and total length $2L$ has a roller support at $A$, a roller support at $B$, a fixed support at $C$ and carries a linearly distributed load as shown in Fig. 8. Determine $(a)$ the vertical reaction force $A_y$ and slope $\theta_A$ at $A$, $(b)$ the vertical reaction force $B_y$ and slope $\theta_B$ at $B$.

Fig. 8. Continuous beam $AC$ carrying a linearly distributed load

**Solution.** The beam is in static equilibrium. Its free-body diagram is shown in Fig. 9, where we note that the beam is statically indeterminate to the second degree.
Using MoI: Treating $B_y$ as an applied unknown concentrated force, we may use superposition technique to first write the loading function $q$ as follows:

$$q = A_y < x > + B_y < x - L > - \frac{w}{2} < x > - \frac{w}{2L} < x > + \frac{w}{2L} < x - L > + w < x - L >$$

Applying the method of integration to this beam, we write

$$V = A_y < x > + B_y < x - L > - \frac{w}{2} < x > + \frac{w}{4L} < x > + \frac{w}{4L} < x - L > + w < x - L >$$

$$M = Ely' = A_y < x >^3 + B_y < x - L >^3 - \frac{w}{12} < x >^3 - \frac{w}{48L} < x >^3 + \frac{w}{48L} < x - L >^3 + \frac{w}{2} < x - L >$$

The boundary conditions of this beam reveal that $y'(2L) = 0$ and $y(2L) = 0$ at $C$, $y(L) = 0$ at $B$, and $y(0) = 0$ at $A$. Imposing these four conditions, in order, we write

$$0 = A_y (2L)^3 + B_y 2L^3 - \frac{w}{6} (2L)^3 + \frac{w}{12} (2L)^3 + \frac{w}{6} L^3 + C_1$$

$$0 = A_y \frac{6}{6} (2L)^3 + B_y \frac{2}{6} L^3 - \frac{w}{12} (2L)^3 - \frac{w}{48L} (2L)^3 + \frac{w}{48L} L^3 + C_1 (2L) + C_2$$

$$0 = A_y \frac{6}{6} L^3 - \frac{w}{48L} L^3 - \frac{w}{24L} L^3 + C_1 L + C_2$$

$$0 = C_2$$

The above four simultaneous equations yield

$$A_y = \frac{39wL}{140} \quad B_y = \frac{31wL}{56} \quad C_1 = -\frac{3wL^3}{140} \quad C_2 = 0$$

Using these values and the foregoing equation for $El y'$, we write

$$\theta_d = \frac{y'|_{x=0}}{E I} = -\frac{3wL^3}{140EI} \quad \theta_b = \frac{y'|_{x=L}}{E I} = \frac{1}{EI} \left( \frac{A_y}{2} L^2 - \frac{w}{12} L^3 - \frac{w}{48L} L^4 + C_1 \right) = \frac{23wL^3}{1680EI}$$

We report that

$$A_y = \frac{39wL}{140} \quad \theta_d = \frac{3wL^3}{140EI} \quad B_y = \frac{31wL}{56} \quad \theta_b = \frac{23wL^3}{1680EI}$$
Using MoMF: Let the method of model formulas be now applied to this beam. We notice that the beam $AC$ has a total length $2L$, which will be the value for the parameter $L$ in all the model formulas in Eqs. (1) through (4). We see that the shear force $V_A$ at left end $A$ is equal to $A_y$, the moment $M_A$ and deflection $y_A$ at $A$ are zero, the deflection $y_B$ at $B$ is zero, and the slope $\theta_C$ and deflection $y_C$ at $C$ are zero. Applying the model formulas in Eqs. (3) and (4) to the beam $AC$ and using Eq. (2) to impose the condition that $y_B = y(L) = 0$ at $B$, in that order, we write

$$0 = \theta_A + \frac{A_y}{2EI}(2L)^2 + 0 - \frac{B_y}{2EI}(2L-L)^2 + 0 - \frac{w}{6EI}(2L)^3 - \frac{w-(w/2)}{24EIL}(2L)^4$$

$$+ \frac{w}{6EI}(2L-L)^3 + \frac{w-(w/2)}{24EIL}(2L-L)^4 + 0 - 0$$

$$0 = 0 + \theta_A(L + \frac{A_y}{6EI})L^3 + 0 - 0 - \frac{w}{24EIL}(2L)^4 - \frac{w-(w/2)}{120EIL}L^5 + 0 + 0 - 0$$

These three simultaneous equations yield

$$A_y = \frac{39wL}{140} \quad \theta_A = \frac{-3wL^3}{140EI} \quad B_y = \frac{31wL}{56}$$

Using these values and applying the model formula in Eq. (1), we write

$$\theta_B = y'_{\mid B} = \theta_A + \frac{A_y}{2EI}L^2 + 0 - 0 + 0 - \frac{w}{6EI}L^3 - \frac{w-(w/2)}{24EIL}L^4 + 0 + 0 - 0 = \frac{23wL^3}{1680EI}$$

We report that

$$A_y = \frac{39wL}{140} \quad \theta_A = \frac{3wL^3}{140EI} \quad B_y = \frac{31wL}{56} \quad \theta_B = \frac{23wL^3}{1680EI}$$

Example 5. A stepped beam $AD$, propped at $A$ and fixed at $D$, carries a concentrated force $P$ at $B$ as shown in Fig. 10, where the segments $AC$ and $CD$ have flexural rigidities $EI_1$ and $EI_2$, respectively. Determine (a) the reaction force $A_y$ at $A$; (b) the slopes $\theta_A$, $\theta_B$, and $\theta_C$ at $A$, $B$, and $C$; (c) the deflections $y_B$ and $y_C$ at $B$ and $C$.

![Fig. 10. Stepped beam AD being supported at A and D and loaded at B](image)

Solution. The beam is in static equilibrium and is statically indeterminate to the first degree. To facilitate the analysis of this beam, we first draw the free-body diagrams of its segments $AC$ and $CD$ as shown in Fig. 11.
Fig. 11  Free-body diagrams of the segments AC and CD

● **Using MoI**: Applying the *method of integration* to the segment AC as shown in Fig. 11(a), we write

\[ q_{AC} = A_y < x >^{-1} - P < x - L >^{-1} \]
\[ V_{AC} = A_y < x >^{0} - P < x - L >^{0} \]
\[ M_{AC} = EI_1 y'_{AC} = A_y < x >^{1} - P < x - L >^{1} \]
\[ EI_1 y''_{AC} = \frac{A_y}{2} < x >^{2} - \frac{P}{2} < x - L >^{2} + C_1 \]
\[ EI_1 y''_{AC} = \frac{A_y}{6} < x >^{3} - \frac{P}{6} < x - L >^{3} + C_1 x + C_2 \]

Applying the *method of integration* to the segment CD as shown in Fig. 11(b), we write

\[ q_{CD} = V_c < x - 2L >^{-1} + M_c < x - 2L >^{-2} \]
\[ V_{CD} = V_c < x - 2L >^{0} + M_c < x - 2L >^{-1} \]
\[ M_{CD} = EI_2 y'_{CD} = V_c < x - 2L >^{1} + M_c < x - 2L >^{0} \]
\[ EI_2 y''_{CD} = \frac{V_c}{2} < x - 2L >^{2} + M_c < x - 2L >^{1} + C_3 \]
\[ EI_2 y''_{CD} = \frac{V_c}{6} < x - 2L >^{3} + \frac{M_c}{2} < x - 2L >^{2} + C_3 x + C_4 \]

The *boundary conditions* of the beam reveal that \( y_{AC}(0) = 0 \) at A; \( y_{AC}(2L) = y_{CD}(2L) \) and \( y''_{AC}(2L) = y''_{CD}(2L) \) at C; \( y_{CD}(3L) = 0 \) and \( y''_{CD}(3L) = 0 \) at D. Imposing these *five* conditions, in order, we write

\[ 0 = C_2 \]  
\[ 0 = \frac{1}{I_2} \left[ \frac{A_y}{6} (2L)^3 - \frac{P}{6} (L)^3 + C_1 (2L) + C_2 \right] = \frac{1}{I_2} \left[ C_1 (2L) + C_4 \right] \]  
\[ 0 = \frac{1}{I_2} \left[ \frac{A_y}{2} (2L)^2 - \frac{P}{2} (L)^2 + C_1 \right] = C_3 \]  
\[ 0 = \frac{V_c}{6} (L)^3 + \frac{M_c}{2} (L)^2 + C_1 (3L) + C_4 \]  
\[ 0 = \frac{V_c}{2} (L)^2 + M_c (L) + C_3 \]

For equilibrium of the segment AC in Fig. 11(a), we write
The above seven simultaneous Eqs. (a) through (g) yield

\[
A_y = \frac{(23 I_1 + 5 I_2) P}{2(19 I_1 + 8 I_2)} \quad \quad V_C = -\frac{(15 I_1 + 11 I_2) P}{2(19 I_1 + 8 I_2)} \quad \quad M_C = \frac{(4 I_1 - 3 I_2)PL}{19 I_1 + 8 I_2}
\]

\[
C_1 = -\frac{(I_1^2 + 31 I_1 I_2 + 4 I_2^2)PL^2}{4 I_2(19 I_1 + 8 I_2)} \quad \quad C_3 = \frac{(-I_1 + 23 I_2)PL^2}{4(19 I_1 + 8 I_2)}
\]

\[
C_2 = 0 \quad \quad C_4 = -\frac{(89 I_2)PL^3}{6(19 I_1 + 8 I_2)}
\]

Using these values and the foregoing equations for \( EI_y'_{AC} \) and \( EI_y_{AC} \) we write

\[
\theta_A = \left. y_{AC}' \right|_{x=a} = \frac{C_1}{EI} = -\frac{(I_1^2 + 31 I_1 I_2 + 4 I_2^2)PL^2}{4EI I_2(19 I_1 + 8 I_2)}
\]

\[
\theta_B = \left. y_{AC}' \right|_{x=L} = \frac{A_y L^2}{2EI} + \frac{C_1}{EI} = -\frac{(I_1^2 + 8 I_1 I_2 - I_2^2)PL^2}{4EI I_2(19 I_1 + 8 I_2)}
\]

\[
\theta_C = \left. y_{AC}' \right|_{x=2L} = \frac{A_y (2L)^2}{2EI} - \frac{PL^2}{2EI} + \frac{C_1}{EI} = \frac{(-I_1 + 23 I_2)PL^2}{4EI I_2(19 I_1 + 8 I_2)}
\]

\[
y_B = \left. y_{AC} \right|_{x=L} = \frac{A_y L^3}{6EI} + \frac{C_1 L}{EI} = -\frac{(3I_1^2 + 70 I_1 I_2 + 7I_2^2)PL^3}{12EI I_2(19 I_1 + 8 I_2)}
\]

\[
y_C = \left. y_{AC} \right|_{x=2L} = \frac{4A_y L^3}{3EI} - \frac{PL^3}{EI} + \frac{2C_1 L}{EI} = -\frac{(3I_1 + 20 I_2)PL^3}{6EI I_2(19 I_1 + 8 I_2)}
\]

We report that

\[
A_y = \frac{(23 I_1 + 5 I_2) P}{2(19 I_1 + 8 I_2)} \quad \quad \theta_A = \frac{(I_1^2 + 31 I_1 I_2 + 4 I_2^2)PL^2}{4EI I_2(19 I_1 + 8 I_2)}
\]

\[
\theta_B = \frac{(I_1^2 + 8 I_1 I_2 - I_2^2)PL^2}{4EI I_2(19 I_1 + 8 I_2)} \quad \quad \theta_C = \frac{(-I_1 + 23 I_2)PL^2}{4EI I_2(19 I_1 + 8 I_2)}
\]

\[
y_B = \frac{(3I_1^2 + 70 I_1 I_2 + 7I_2^2)PL^3}{12EI I_2(19 I_1 + 8 I_2)} \quad \quad y_C = \frac{(3I_1 + 20 I_2)PL^3}{6EI I_2(19 I_1 + 8 I_2)}
\]

**Using MoMF:** Let the method of model formulas be now applied to this stepped beam. We first divide the beam into two segments, whose free-body diagrams are shown in parts (a) and (b) of Fig. 11. In particular, note that the segment \( AC \) has a total length \( 2L \), which will be the value for the parameter \( L \) in all the model formulas in Eqs. (1) through (4). We see that shear force
\( V_A \) at the left end \( A \) is equal to \( A_y \), the moment \( M_A \) and deflection \( y_A \) at \( A \) are zero. We may let the slope and deflection at \( C \) be \( \theta_c \) and \( y_c \), respectively. Applying the model formulas in Eqs. (3) and (4) to segment \( AC \) in Fig. 11(a), in that order, we write

\[
\theta_c = \theta_A + \frac{A_y(2L)^2}{2EI_1} + 0 - \frac{P}{2EI_1}(2L - L)^2 + 0 - 0 + 0 + 0 + 0 = 0
\]  
\[
y_c = 0 + \theta_A(2L) + \frac{A_y(2L)}{6EI_1} + 0 - \frac{P}{6EI_1}(2L - L)^3 + 0 - 0 + 0 + 0 + 0 = 0
\]

For equilibrium of the segment \( AC \) in Fig. 11(a), we write

\[ + \sum F_y = 0: \quad A_y - V_c - P = 0 \]  
\[ + \sum M_c = 0: \quad -2LA_y + LP + M_c = 0 \]

We note that the slope \( \theta_c \) and deflection \( y_c \) at end \( D \) of segment \( CD \) are zero. Applying the model formulas in Eqs. (3) and (4) to segment \( CD \) in Fig. 11(b), in that order, we write

\[
0 = \theta_c + \frac{V_cL^2}{2EI_2} + \frac{M_cL}{EI_2} - 0 + 0 - 0 + 0 + 0 + 0 = 0
\]  
\[
0 = y_c + \theta_cL + \frac{V_cL^3}{6EI_2} + \frac{M_cL^2}{2EI_2} - 0 + 0 - 0 + 0 + 0 + 0 = 0
\]

The above six simultaneous Eqs. (h) through (m) yield

\[
A_y = \frac{(23I_1 + 5I_2)P}{2(19I_1 + 8I_2)} \quad V_c = -\frac{(15I_1 + 11I_2)P}{2(19I_1 + 8I_2)} \quad M_c = \frac{(4I_1 - 3I_2)PL}{19I_1 + 8I_2}
\]  
\[
\theta_A = -\frac{(I_1^2 + 31I_1I_2 + 4I_2^2)PL^2}{4EI_1L^2(19I_1 + 8I_2)} \quad \theta_c = \frac{(-I_1 + 23I_2)PL^2}{4EI_2(19I_1 + 8I_2)} \quad y_c = -\frac{(3I_1 + 20I_2)PL^3}{6EI_2(19I_1 + 8I_2)}
\]

Using these values and applying the model formulas in Eqs. (1) and (2), we write

\[
\theta_B = y_{acj}^{j} \bigg|_{\gamma = L} = \theta_A + \frac{A_y}{2EI_1}L^2 + 0 - 0 + 0 - 0 + 0 + 0 + 0 = -\frac{(I_1^2 + 8I_1I_2 - I_2^2)PL^2}{4EI_1L^2(19I_1 + 8I_2)}
\]  
\[
y_B = y_{acj}^{j} \bigg|_{\gamma = L} = 0 + \theta_A + \frac{A_y}{6EI_1}L^3 + 0 - 0 + 0 - 0 + 0 + 0 + 0 + 0 = -\frac{(3I_1^2 + 70I_1I_2 + 7I_2^2)PL^3}{12EI_1L^2(19I_1 + 8I_2)}
\]

We report that

\[
A_y = \frac{(23I_1 + 5I_2)P}{2(19I_1 + 8I_2)} \quad \theta_A = \frac{(I_1^2 + 31I_1I_2 + 4I_2^2)PL^2}{4EI_1L^2(19I_1 + 8I_2)} \quad y_B = \frac{(3I_1^2 + 70I_1I_2 + 7I_2^2)PL^3}{12EI_1L^2(19I_1 + 8I_2)}
\]  
\[
\theta_B = \frac{(I_1^2 + 8I_1I_2 - I_2^2)PL^2}{4EI_1L^2(19I_1 + 8I_2)} \quad \theta_c = \frac{(-I_1 + 23I_2)PL^2}{4EI_2(19I_1 + 8I_2)} \quad y_c = \frac{(3I_1 + 20I_2)PL^3}{6EI_2(19I_1 + 8I_2)}
\]
• **Checking Obtained Results:** The effort to obtain the solution for the problem in this example is algebraically challenging. Naturally, it is desirable to check the preceding obtained results against a known solution for the special case of

\[
I_1 = I_2 = 1
\]

For such a special case, the preceding obtained results degenerate into the following:

\[
\begin{align*}
\theta_a &= \frac{14PL}{27} \uparrow \\
\theta_c &= \frac{11PL^2}{54EI} \uparrow \\
\theta_b &= \frac{2PL^2}{27EI} \downarrow \\
y_b &= \frac{20PL^3}{81EI} \downarrow \\
y_c &= \frac{23PL^3}{162EI} \downarrow
\end{align*}
\]

We find that these special results are indeed **consistent** with those given at the end of textbooks.\(^9\)

### III. An Effective Approach to Teaching the MoMF

The *method of model formulas* is a general methodology that employs a set of *four equations* to serve as *model formulas* in solving problems involving statically indeterminate reactions, as well as slopes and deflections, of elastic beams. The first two model formulas are for the slope and deflection at any position \(x\) of the beam and contain rudimentary singularity functions, while the other two model formulas contain only traditional algebraic expressions. Generally, this method requires much less effort in solving beam deflection problems. Most students favor this method because they can solve problems in shorter time using this method and they score higher in tests.

The five examples, arranged in order of increasing challenge, in Section II provide a variety of head-to-head comparisons between solutions by the traditional *method of integration* and those by the *method of model formulas*; and all of the solutions are, respectively, in agreement. Thus, all solutions by the *method of model formulas* are naturally correct. Experience shows that the following steps form a pedagogy that can be used to effectively introduce and teach the *method of model formulas* to students to enrich their study and set of skills in determining statically indeterminate reactions and deflections of elastic beams in mechanics of materials:

- Teach the traditional *method of integration* and the imposition of boundary conditions.
- Teach the rudiments of *singularity functions* and utilize them in the *method of integration*.
- Go over briefly the derivation\(^1\) of the *four model formulas* in terms of *singularity functions*.
- Give students the heads-up on the following advantages in the *method of model formulas*:
  - No need to integrate or evaluate constants of integration.
  - Not prone to generate a large number of simultaneous equations **even if**
    - the beam carries multiple concentrated loads (forces or moments),
    - the beam has one or more simple supports *not* at its ends,
    - the beam has linearly distributed loads *not* starting at its left end, and
    - the beam has linearly distributed loads *not* ending at its right end.
- Demonstrate solutions of several beam problems by the *method of model formulas*.
- Assess the solutions obtained (e.g., comparing with solutions by another method).
Although solutions obtained by the method of model formulas are often more direct than those obtained by the method of integration, a one-page excerpt from the method of model formulas, such as that shown in Fig. 1, must be made available to those who used this method. Still, one may remember that a list of formulas for slope and deflection of selected beams having a variety of supports and loading is also needed by persons who use the method of superposition. In this regard, the method of model formulas is on a par with the method of superposition.

IV. Concluding Remarks

In the method of model formulas, no explicit integration or differentiation is involved in applying any of the model formulas. The model formulas essentially serve to provide material equations (which involve and reflect the material property) besides the equations of static equilibrium of the beam that can readily be written. Selected applied loads are illustrated in Fig. 1(a), which cover most of the loads encountered in undergraduate Mechanics of Materials. In the case of a nonlinearly distributed load on the beam, the model formulas may be modified by the user for such a load.

The method of model formulas is best taught to students as an alternative method, after they have learned one or more of the traditional methods.\textsuperscript{2-12} This new method enriches students’ study and set of skills in determining reactions and deflections of beams, and it provides engineers with a means to independently check their solutions obtained using traditional methods.

References

ING-CHANG JONG
Ing-Chang Jong is Professor of Mechanical Engineering at the University of Arkansas. He received his BSCE in 1961 from the National Taiwan University, his MSCE in 1963 from South Dakota School of Mines and Technology, and his Ph.D. in Theoretical and Applied Mechanics in 1965 from Northwestern University. Dr. Jong was Chair of the Mechanics Division, ASEE, in 1996-97. He received the Archie Higdon Distinguished Educator Award in 2009.

WILLIAM T. SPRINGER
William T. Springer is Associate Professor of Mechanical Engineering at the University of Arkansas. He received his BSME in 1974 from the University of Texas at Arlington, his MSME in 1979 from the University of Texas at Arlington, and his Ph.D. in Mechanical Engineering in 1982 from the University of Texas at Arlington. Dr. Springer is active in ASME where he received the Dedicated Service Award in 2006 and attained Fellow Grade in 2008.

RICK J. COUVILLION
Rick J. Couvillion is Associate Professor of Mechanical Engineering at the University of Arkansas. He received his BSME in 1975 from the University of Arkansas. After 2½ years in industry, he returned to school, received his MSME from Georgia Tech in 1978, and his Ph.D. from Georgia Tech in 1981. He is active in ASHRAE and was elected an ASME fellow in 2004. He was chosen for the University of Arkansas Teaching Academy in 2005.