AC 2011-91: TEACHING DEFLECTIONS OF BEAMS: COMPARISON OF ADVANTAGES OF METHOD OF MODEL FORMULAS VERSUS METHOD OF SUPERPOSITION

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Teaching Deflections of Beams: Comparison of Advantages of Method of Model Formulas versus Method of Superposition

Abstract

The *method of model formulas* is a new method for solving statically indeterminate reactions and deflections of elastic beams. Since its publication in the *IJEE* in 2009, instructors of Mechanics of Materials have considerable interest in teaching this method to enrich students' set of skills in determining beam reactions and deflections. Besides, instructors are interested in seeing relative advantages of this method versus the traditional *method of superposition*. This paper is aimed at comparing the method of model formulas with the method of superposition regarding (*a*) their methodology and pedagogy, (*b*) the availability of a one-page excerpt from the method of model formulas, (*c*) the availability of a one-page collection of deflection formulas of selected beams for the method of superposition, and (*d*) assessment of their effectiveness in solving problems of reactions and deflections of beams in several identical given problems.

I. Introduction

Beams are longitudinal members subjected to transverse loads. Students usually first learn the design of beams for strength. Then they learn the determination of deflections of beams under a variety of loads. Methods used in determining statically indeterminate reactions and deflections of elastic beams include:[2-13] method of integration (*with* or *without* use of singularity functions), method using moment-area theorems, method of conjugate beam, method using Castigliano's theorem, method of superposition, method of segments, and method of model formulas.

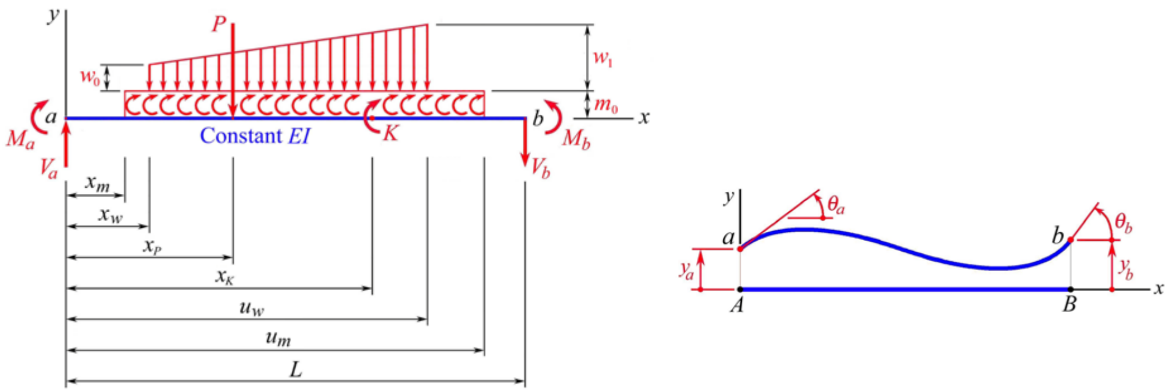
The *method of model formulas*[1] is a newly propounded method. Beginning with an elastic beam under a selected preset general loading, a set of *four* model formulas are derived and established for use in this new method. These four formulas are expressed in terms of the following:

(*a*) flexural rigidity of the beam;
(*b*) slopes, deflections, shear forces, and bending moments at both ends of the beam;
(*c*) typical applied loads (concentrated force, concentrated moment, linearly distributed force, and uniformly distributed moment) *somewhere* on the beam.

For starters, one must note that a working proficiency in the rudiments of *singularity functions* is a **prerequisite** to using the *method of model formulas*. To benefit a wider readership, which may have different specialties in mechanics, and to avoid or minimize any possible misunderstanding, this paper includes summaries of the rudiments of singularity functions and the sign conventions for beams. Readers, who are familiar with these topics, may skip the summaries. An excerpt from the *method of model formulas* is needed and shown in Fig. 1, courtesy of *IJEE*.[1] Besides, a collection of slope and deflection formulas of selected beams for the *method of superposition* is needed and shown in Fig. 2, courtesy of a textbook by S. Timoshenko and G. H. MacCullough.[2]

Excerpt from the **Method of Model Formulas**

Courtesy: *Int. J. Engng. Ed.*, Vol. 25, No. 1, pp. 65-74, 2009



(*a*) (*b*)

Positive directions of forces, moments, slopes, and deflections

$$\begin{aligned} y' = {} & \theta_a + \frac{V_a}{2EI}x^2 + \frac{M_a}{EI}x - \frac{P}{2EI}\langle x - x_P\rangle^2 + \frac{K}{EI}\langle x - x_K\rangle^1 - \frac{w_0}{6EI}\langle x - x_w\rangle^3 \\ & - \frac{w_1 - w_0}{24EI(u_w - x_w)}\langle x - x_w\rangle^4 + \frac{w_1}{6EI}\langle x - u_w\rangle^3 + \frac{w_1 - w_0}{24EI(u_w - x_w)}\langle x - u_w\rangle^4 \\ & + \frac{m_0}{2EI}\langle x - x_m\rangle^2 - \frac{m_0}{2EI}\langle x - u_m\rangle^2 \end{aligned} \tag{1}$$

$$\begin{aligned} y = {} & y_a + \theta_a x + \frac{V_a}{6EI}x^3 + \frac{M_a}{2EI}x^2 - \frac{P}{6EI}\langle x - x_P\rangle^3 + \frac{K}{2EI}\langle x - x_K\rangle^2 - \frac{w_0}{24EI}\langle x - x_w\rangle^4 \\ & - \frac{w_1 - w_0}{120EI(u_w - x_w)}\langle x - x_w\rangle^5 + \frac{w_1}{24EI}\langle x - u_w\rangle^4 + \frac{w_1 - w_0}{120EI(u_w - x_w)}\langle x - u_w\rangle^5 \\ & + \frac{m_0}{6EI}\langle x - x_m\rangle^3 - \frac{m_0}{6EI}\langle x - u_m\rangle^3 \end{aligned} \tag{2}$$

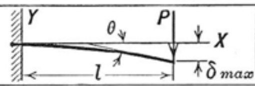
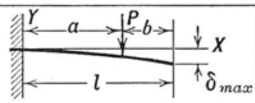
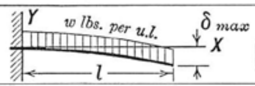
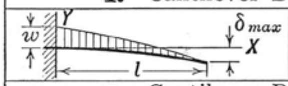
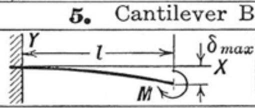
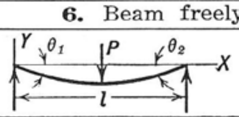
$$\begin{aligned} \theta_b = {} & \theta_a + \frac{V_a L^2}{2EI} + \frac{M_a L}{EI} - \frac{P}{2EI}(L - x_P)^2 + \frac{K}{EI}(L - x_K) - \frac{w_0}{6EI}(L - x_w)^3 \\ & - \frac{w_1 - w_0}{24EI(u_w - x_w)}(L - x_w)^4 + \frac{w_1}{6EI}(L - u_w)^3 + \frac{w_1 - w_0}{24EI(u_w - x_w)}(L - u_w)^4 \\ & + \frac{m_0}{2EI}(L - x_m)^2 - \frac{m_0}{2EI}(L - u_m)^2 \end{aligned} \tag{3}$$

$$\begin{aligned} y_b = {} & y_a + \theta_a L + \frac{V_a L^3}{6EI} + \frac{M_a L^2}{2EI} - \frac{P}{6EI}(L - x_P)^3 + \frac{K}{2EI}(L - x_K)^2 - \frac{w_0}{24EI}(L - x_w)^4 \\ & - \frac{w_1 - w_0}{120EI(u_w - x_w)}(L - x_w)^5 + \frac{w_1}{24EI}(L - u_w)^4 + \frac{w_1 - w_0}{120EI(u_w - x_w)}(L - u_w)^5 \\ & + \frac{m_0}{6EI}(L - x_m)^3 - \frac{m_0}{6EI}(L - u_m)^3 \end{aligned} \tag{4}$$

Fig. 1. Model loading and beam deflection formulas for the *method of model formulas*

Deflection Formulas of Selected Beams for the **Method of Superposition**

Courtesy: Timoshenko and MacCullough, *Elements of Strength of Materials*, *3th Ed.*, pp. 182-183, D. Van Nostrand Company, Inc., 1949

ART. 59	Slope at free end.	Deflection at any section in terms of x: δ is positive downward.	Maximum deflection.
1. Cantilever Beam — Concentrated load P at the free end.			
	$\theta=\frac{Pl^2}{2EI}$	$\delta=\frac{Px^2}{6EI}(3l-x)$	$\delta_{max}=\frac{Pl^3}{3EI}$
2. Cantilever Beam — Concentrated load P at any point.			
	$\theta=\frac{Pa^2}{2EI}$	$\delta=\frac{Px^2}{6EI}(3a-x)$ for $0<x<a$. $\delta=\frac{Pa^2}{6EI}(3x-a)$ for $a<x<l$	$\delta_{max}=\frac{Pa^2}{6EI}(3l-a)$
3. Cantilever Beam — Uniformly distributed load of w lbs. per unit length.			
	$\theta=\frac{wl^3}{6EI}$	$\delta=\frac{wx^2}{24EI}(x^2+6l^2-4lx)$	$\delta_{max}=\frac{wl^4}{8EI}$
4. Cantilever Beam — Uniformly varying load; maximum intensity w lbs. per unit length.			
	$\theta=\frac{wl^3}{24EI}$	$\delta=\frac{wx^2}{120lEI}(10l^3-10l^2x+5lx^2-x^3)$	$\delta_{max}=\frac{wl^4}{30EI}$
5. Cantilever Beam — Couple M applied at the free end.			
	$\theta=\frac{Ml}{EI}$	$\delta=\frac{Mx^2}{2EI}$	$\delta_{max}=\frac{Ml^2}{2EI}$
6. Beam freely supported at ends — Concentrated load P at the center.			
	$\theta_1=\theta_2=\frac{Pl^2}{16EI}$	$\delta=\frac{Px}{12EI}\left(\frac{3l^2}{4}-x^2\right)$ for $0<x<\frac{l}{2}$	$\delta_{max}=\frac{Pl^3}{48EI}$

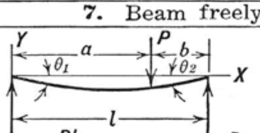
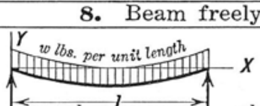
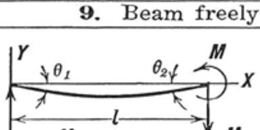
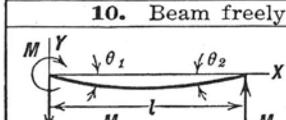
ART. 59	Slope at ends.	Deflection at any section in terms of x: δ is positive downward.	Maximum and center deflections.
7. Beam freely supported at the ends — Concentrated load at any point.			
 $R_1=\frac{Pb}{l}$ $R_2=\frac{Pa}{l}$	Left End. $\theta_1=\frac{Pb(l^2-b^2)}{6lEI}$ Right End. $\theta_2=\frac{Pab(2l-b)}{6lEI}$	To the left of load P: $\delta=\frac{Pbx}{6lEI}(l^2-x^2-b^2)$ To the right of load P: $\delta=\frac{Pb}{6lEI}\left[\frac{l}{b}(x-a)^3+(l^2-b^2)x-x^3\right]$	$\delta_{max}=\frac{Pb(l^2-b^2)^{3/2}}{9\sqrt{3}\,lEI}$ at $x=\sqrt{\frac{l^2-b^2}{3}}$ At center, if $a>b$ $\delta=\frac{Pb}{48EI}(3l^2-4b^2)$
8. Beam freely supported at the ends — Uniformly distributed load of w lbs. per unit length.			
 $R_1=\frac{wl}{2}$ $R_2=\frac{wl}{2}$	$\theta_1=\theta_2=\frac{wl^3}{24EI}$	$\delta=\frac{wx}{24EI}(l^3-2lx^2+x^3)$	$\delta_{max}=\frac{5wl^4}{384EI}$
9. Beam freely supported at the ends — Couple M at the right end.			
 $R_1=\frac{M}{l}$ $R_2=\frac{M}{l}$	$\theta_1=\frac{Ml}{6EI}$ $\theta_2=\frac{Ml}{3EI}$	$\delta=\frac{Mlx}{6EI}\left(1-\frac{x^2}{l^2}\right)$	$\delta_{max}=\frac{Ml^2}{9\sqrt{3}EI}$ at $x=\frac{l}{\sqrt{3}}$ At center $\delta=\frac{Ml^2}{16EI}$
10. Beam freely supported at the ends — Couple M at the left end.			
 $R_1=\frac{M}{l}$ $R_2=\frac{M}{l}$	$\theta_1=\frac{Ml}{3EI}$ $\theta_2=\frac{Ml}{6EI}$	$\delta=\frac{Mx}{6lEI}(l-x)(2l-x)$	$\delta_{max}=\frac{Ml^2}{9\sqrt{3}EI}$ at $x=\left(1-\frac{1}{\sqrt{3}}\right)l$ At center $\delta=\frac{Ml^2}{16EI}$

Fig. 2. Slope and deflection formulas of selected beams for the *method of superposition*

Summary of rudiments of singularity functions

Notice that the argument of a singularity function is enclosed by angle brackets (*i.e.*, < >). The argument of a regular function continues to be enclosed by parentheses [*i.e.*, ( )]. The rudiments of singularity functions include the following:[1,8,9]

$$< x-a >^n = (x-a)^n \quad \text{if} \quad x-a \geq 0 \quad \text{and} \quad n > 0 \tag{5}$$

$$< x-a >^n = 1 \quad \text{if} \quad x-a \geq 0 \quad \text{and} \quad n = 0 \tag{6}$$

$$< x-a >^n = 0 \quad \text{if} \quad x-a < 0 \quad \text{or} \quad n < 0 \tag{7}$$

$$\int_{-\infty}^{x} < x-a >^n dx = \frac{1}{n+1} < x-a >^{n+1} \quad \text{if} \quad n > 0 \tag{8}$$

$$\int_{-\infty}^{x} < x-a >^n dx = < x-a >^{n+1} \quad \text{if} \quad n \leq 0 \tag{9}$$

$$\frac{d}{dx} < x-a >^n = n < x-a >^{n-1} \quad \text{if} \quad n > 0 \tag{10}$$

$$\frac{d}{dx} < x-a >^n = < x-a >^{n-1} \quad \text{if} \quad n \leq 0 \tag{11}$$

Equations (6) and (7) imply that, in using singularity functions for beams, we take

$$b^0 = 1 \quad \text{for} \quad b \geq 0 \tag{12}$$

$$b^0 = 0 \quad \text{for} \quad b < 0 \tag{13}$$

Summary of sign conventions for beams

In the *method of model formulas*, the adopted sign conventions for various model loadings on the beam and for deflections of the beam with a constant flexural rigidity *EI* are illustrated in Fig. 1. Notice the following key points:

- A *shear force* is *positive* if it acts upward on the left (or downward on the right) face of the beam element [e.g., $\mathbf{V}_a$ at the left end *a*, and $\mathbf{V}_b$ at the right end *b* in Fig. 1(*a*)].
- At ends of the beam, a *moment* is *positive* if it tends to cause compression in the top fiber of the beam [e.g., $\mathbf{M}_a$ at the left end *a*, and $\mathbf{M}_b$ at the right end *b* in Fig. 1(*a*)].
- If not at ends of the beam, a *moment* is *positive* if it tends to cause compression in the top fiber of the beam just to the right of the position where it acts [e.g., the concentrated moment $\mathbf{K} = K \circlearrowleft$ and the uniformly distributed moment with intensity m_0 in Fig. 1(*a*)].
- A *concentrated force* or a *distributed force* applied to the beam is *positive* if it is directed downward [e.g., the concentrated force $\mathbf{P} = P \downarrow$, the linearly distributed force with intensity w_0 on the left side and intensity w_1 on the right side in Fig. 1(*a*), where the distribution becomes uniform if $w_0 = w_1$].

The slopes and deflections of a beam displaced from *AB* to *ab* are shown in Fig. 1(*b*). Note that

- A *positive slope* is a counterclockwise angular displacement [e.g., θ_a and θ_b in Fig. 1(*b*)].
- A *positive deflection* is an upward linear displacement [e.g., y_a and y_b in Fig. 1(*b*)].

■ Methodology and pedagogy of the method of model formulas

The four model formulas in Eqs. (1) through (4) were derived in great detail in the paper that propounded the *method of model formulas*.[1] For convenience of readers, let us take a brief overview of how these model formulas are obtained. Basically, it starts out with the loading function q, written in terms of singularity functions for the beam ab in Fig. 1; as follows:

$$\begin{aligned} q = {} & V_a < x >^{-1} + M_a < x >^{-2} - P < x - x_P >^{-1} + K < x - x_K >^{-2} - w_0 < x - x_w >^0 \\ & - \frac{w_1 - w_0}{u_w - x_w} < x - x_w >^1 + w_1 < x - u_w >^0 + \frac{w_1 - w_0}{u_w - x_w} < x - u_w >^1 \\ & + m_0 < x - x_m >^{-1} - m_0 < x - u_m >^{-1} \end{aligned} \tag{14}$$

By integrating q, one can write the shear force V and the bending moment M for the beam ab in Fig. 1. Letting the flexural rigidity of the beam ab be EI, y be the deflection, y' be the slope, and y'' be the second derivative of y with respect to the abscissa x, which defines the position of the section along the axis of the beam under consideration, one may apply the relation $EIy'' = M$ and readily obtain the expressions for EIy' and EIy via integration. The slope and deflection of the beam are θ_a and y_a at its left end a (*i.e.*, at x = 0), and are θ_b and y_b at the right end b (*i.e.*, at $x = L$), as illustrated in Fig. 1. Imposition of these boundary conditions will yield the four model formulas in Eqs. (1) through (4).

Note that L in the model formulas in Eqs. (1) through (4) is a *parameter* representing the *total length* of the beam segment. In other words, this L is to be replaced by the *total length* of the beam segment to which the model formulas are applied. Furthermore, notice that this method allows one to treat reactions at interior supports (i.e., those *not* at the ends of the beam) as applied concentrated forces or moments, as appropriate. All one has to do is to simply impose the additional boundary conditions at the points of interior supports for the beam segment. Thus, statically indeterminate reactions as well as slopes and deflections of beams can be determined.

A beam needs to be divided into segments for analysis only if (*a*) it is a combined beam (*e.g.*, a *Gerber beam*) having discontinuities in slope at hinge connections between segments, and (*b*) it contains segments with different flexural rigidities (*e.g.*, a stepped beam).

■ Methodology and pedagogy of the method of superposition

The *method of superposition* for the deflection of beams is a traditional method that can be found in most textbooks on mechanics of materials.[2-8] The methodology and pedagogy of this method may not require a detailed description in this paper.

Basically, this method requires that a table containing a good collection of slope and deflection formulas of selected beams, such as the one shown in Fig. 2, be available. In this method, the resulting deflection of a beam due to the various loads applied on the beam is taken to be the same as the sum of the deflections of the beam due to individual loads applied *one at a time* on the beam. However, this method will *fail* to work if the table lacks formulas for certain types of load (*e.g.*, concentrated moment acting at other than an end point of a *simply supported beam*).

II. Teaching and Learning a New Method via Contrast between Solutions

Mechanics is mostly a deductive science, but learning is mostly an inductive process. For the purposes of *teaching* and *learning*, all examples will **first** be solved by the new *method of model formulas* (**MoMF**). **Then** the same problems in the examples will be solved by the traditional *method of superposition* (**MoS**) — unless impossible — using ***just*** the formulas for slopes and deflections of beams that are shown in Fig. 2. To stay within a limit in this paper, formulas that are available from other sources will not be admitted in the solution using **MoS** in the examples.

Example 1. A simply supported beam *AD* with constant flexural rigidity *EI* and total length *L* is acted on by a concentrated force $P\downarrow$ at *B* and a concentrated moment $PL \circlearrowleft$ at *C* as shown in Fig. 3. Determine (*a*) the slopes θ_A and θ_D at *A* and *D*, respectively; (*b*) the deflection y_B at *B*.

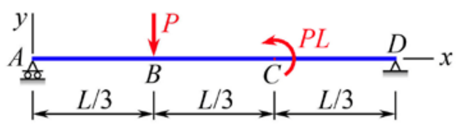


Fig. 3. Simply supported beam *AD* carrying concentrated loads

Solution. The beam is statically determinate. Its free-body diagram is shown in Fig. 4.

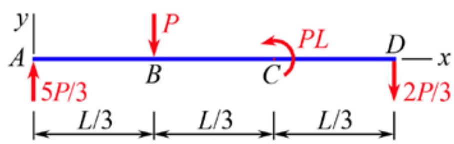


Fig. 4. Free-body diagram of the simply supported beam *AD*

● **Using MoMF:** In applying the *method of model formulas* to this beam, we must adhere to the sign conventions as illustrated in Fig. 1. At the left end *A*, the moment M_A is 0, the shear force V_A is 5*P*/3, the deflection y_A is 0, but the slope θ_A is unknown. At the right end *D*, the deflection y_D is 0, but the slope θ_D is unknown. Note in the model formulas that we have $x_P = L/3$ for the concentrated force $P\downarrow$ at *B* and $x_K = 2L/3$ for the concentrated moment $PL \circlearrowleft$ at *C*. Applying the model formulas in Eqs. (3) and (4), successively, to this beam *AD*, we write

$$\theta_D = \theta_A + \frac{(5P/3)L^2}{2EI} + 0 - \frac{P}{2EI}\left(L - \frac{L}{3}\right)^2 + \frac{-PL}{EI}\left(L - \frac{2L}{3}\right) - 0 - 0 + 0 + 0 + 0 - 0$$

$$0 = 0 + \theta_A L + \frac{(5P/3)L^3}{6EI} + 0 - \frac{P}{6EI}\left(L - \frac{L}{3}\right)^3 + \frac{-PL}{2EI}\left(L - \frac{2L}{3}\right)^2 - 0 - 0 + 0 + 0 + 0 - 0$$

These *two* simultaneous equations yield

$$\theta_A = -\frac{14PL^2}{81EI} \qquad \theta_D = \frac{17PL^2}{162EI}$$

Using the value of θ_A and applying the model formula in Eq. (2), we write

$$y_B = y\big|_{x=L/3} = 0 + \theta_A\left(\frac{L}{3}\right) + \frac{5P/3}{6EI}\left(\frac{L}{3}\right)^3 + 0 - 0 + 0 - 0 - 0 + 0 + 0 + 0 - 0 = -\frac{23PL^3}{486EI}$$

We report that

$$\theta_A = \frac{14PL^2}{81EI} \circlearrowright \qquad \theta_D = \frac{17PL^2}{162EI} \circlearrowleft \qquad y_B = \frac{23PL^3}{486EI} \downarrow$$

● **Using MoS:** The beam in Fig. 3 carries a concentrated moment $PL \circlearrowleft$ at *C*, which is not at the end of the beam. Since the table of formulas in Fig. 2 *lacks* this type of load applied at a point other than the end of the simply supported beam, the problem in this example is **prevented from being solved** by the *method of superposition* and the use of ***just*** formulas in Fig. 2.

Assessment of effectiveness. In this example, we see that the *method of model formulas* enables one to directly and successfully obtain the solutions. The *method of superposition* could not be employed to solve the problem in this example solely because there are **no** formulas in Fig. 2 for the slope and deflection of a *simply supported beam* acted on by a concentrated moment at a point other than the end of the beam. This shows a downside of the **MoS** when formulas in the list are inadequate. Thus, the **MoMF** appears to be more general and effective than the **MoS**.

Example 2. A cantilever beam *AC* with constant flexural rigidity *EI* and total length *L* is loaded with a distributed load of intensity *w* in segment *AB* as shown in Fig. 5. Determine (*a*) the slope θ_A and deflection y_A at *A*, (*b*) the slope θ_B and deflection y_B at *B*.

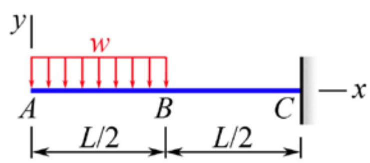


Fig. 5. Cantilever beam *AC* loaded with a distributed load

Solution. The beam is statically determinate. Its free-body diagram is shown in Fig. 6.

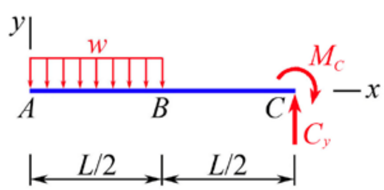


Fig. 6. Free-body diagram of the cantilever beam *AC*

● **Using MoMF:** In applying the *method of model formulas* to solve the problem, we note that the shear force V_A and the bending moment M_A at the free end *A*, as well as the slope θ_C and the deflection y_C at the fixed end *C*, are all zero. Seeing that the uniformly distributed load has $x_w = 0$ and $u_w = L/2$, we apply the model formulas in Eqs. (3) and (4) to the entire beam to write

$$0 = \theta_A + 0 + 0 - 0 + 0 - \frac{w}{6EI}L^3 - 0 + \frac{w}{6EI}\left(L - \frac{L}{2}\right)^3 + 0 + 0 - 0$$

$$0 = y_A + \theta_A L + 0 + 0 - 0 + 0 - \frac{w}{24EI}L^4 - 0 + \frac{w}{24EI}\left(L - \frac{L}{2}\right)^4 + 0 + 0 - 0$$

These *two* simultaneous equations yield

$$\theta_A = \frac{7wL^3}{48EI} \qquad\qquad y_A = -\frac{41wL^4}{384EI}$$

Using these values and applying the model formulas in Eqs. (1) and (2), respectively, we write

$$\theta_B = y'\big|_{x=L/2} = \theta_A + 0 + 0 - 0 + 0 - \frac{w}{6EI}\left(\frac{L}{2}\right)^3 - 0 + 0 + 0 + 0 - 0 = \frac{wL^3}{8EI}$$

$$y_B = y\big|_{x=L/2} = y_A + \theta_A\left(\frac{L}{2}\right) + 0 + 0 - 0 + 0 - \frac{w}{24EI}\left(\frac{L}{2}\right)^4 - 0 + 0 + 0 + 0 - 0 = -\frac{7wL^4}{192EI}$$

We report that

$$\theta_A = \frac{7wL^3}{48EI} \circlearrowleft \qquad y_A = \frac{41wL^4}{384EI} \downarrow \qquad \theta_B = \frac{wL^3}{8EI} \circlearrowleft \qquad y_B = \frac{7wL^4}{192EI} \downarrow$$

● **Using MoS:** For applying the *method of superposition* and adapting to the formulas in Fig. 2, we may *turn* the original given beam in Fig. 5 about a vertical axis through 180° and *note* that it is equivalent to the superposition (*or* sum) of two differently loaded beams as shown in Fig. 7.

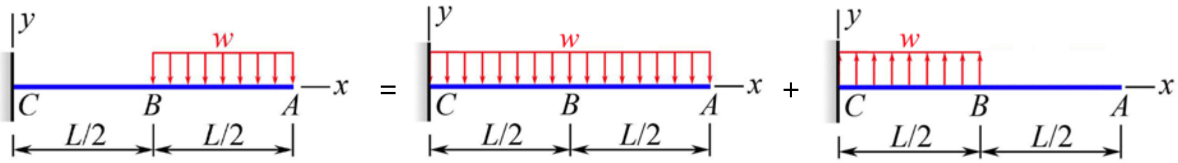


Fig. 7. Original given beam is equivalent to the superposition of two beams shown

Referring to the formulas for the 3rd beam in Fig. 2 and applying the **MoS** to the beam, we write

$$\theta_A = \frac{wL^3}{6EI} - \frac{w(L/2)^3}{6EI} = \frac{7wL^3}{48EI} \qquad \theta_A = \frac{7wL^3}{48EI} \circlearrowleft$$

$$y_A = -\frac{wL^4}{8EI} + \left[\frac{w(L/2)^4}{8EI} + \frac{L}{2}\cdot\left(\frac{w(L/2)^3}{6EI}\right)\right] = -\frac{41wL^4}{384EI} \qquad y_A = \frac{41wL^4}{384EI} \downarrow$$

Referring to the formulas for the 3rd beam in Fig. 2 again, we write

$$y = -\delta = -\frac{w}{24EI}\left(x^4 + 6L^2x^2 - 4Lx^3\right) \qquad y' = \frac{dy}{dx} = -\frac{w}{6EI}\left(x^3 + 3L^2x - 3Lx^2\right)$$

$$\theta_B = -\frac{w}{6EI}\left(x^3 + 3L^2x - 3Lx^2\right)\Big|_{x=L/2} + \frac{w(L/2)^3}{6EI} = \frac{wL^3}{8EI} \qquad \theta_B = \frac{wL^3}{8EI} \circlearrowleft$$

$$y_B = -\frac{w}{24EI}\left(x^4 + 6L^2x^2 - 4Lx^3\right)\Big|_{x=L/2} + \frac{w(L/2)^4}{8EI} = -\frac{7wL^4}{192EI} \qquad y_B = \frac{7wL^4}{192EI} \downarrow$$

Assessment of effectiveness. In this example, we see that the **MoMF** enables us to directly and successfully obtain the solutions, while the **MoS** requires some rotations and combinations of beams to eventually arrive at the same solutions. The **MoMF** is more straightforward than the **MoS**; but they are *about* equally effective in solving the problem in this example.

Example 3. A cantilever beam AC with constant flexural rigidity EI and total length $2L$ is propped at A and carries a concentrated moment M_0 ↺ at B as shown in Fig. 8. Determine (*a*) the vertical reaction force $\mathbf{A}_y$ and slope θ_A at A, (*b*) the slope θ_B and deflection y_B at B.

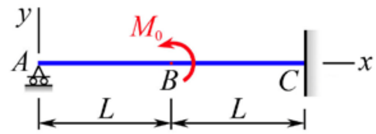


Fig. 8. Cantilever beam AC propped at A and carrying a concentrated moment at B

Solution. The free-body diagram of the beam is shown in Fig. 9, where we note that the beam is statically indeterminate to the *first* degree.

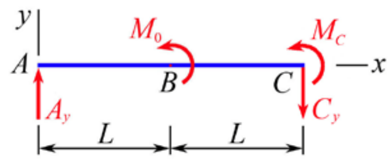


Fig. 9. Free-body diagram of the propped cantilever beam AC

● **Using MoMF:** In applying the *method of model formulas* to this beam, we first note that this beam has a total length of $2L$, which will be the value for the *parameter* L in all of the model formulas in Eqs. (1) through (4). We also note that the deflection y_C and the slope θ_C at C, as well as the deflection y_A at A, are all equal to zero. Applying the model formulas in Eqs. (3) and (4) to this beam, we write

$$0 = \theta_A + \frac{A_y(2L)^2}{2EI} + 0 - 0 + \frac{-M_0}{EI}(2L - L) - 0 - 0 + 0 + 0 + 0 - 0$$

$$0 = 0 + \theta_A(2L) + \frac{A_y(2L)^3}{6EI} + 0 - 0 + \frac{-M_0}{2EI}(2L - L)^2 - 0 - 0 + 0 + 0 + 0 - 0$$

These *two* simultaneous equations yield

$$A_y = \frac{9M_0}{16L} \qquad \theta_A = -\frac{M_0 L}{8EI}$$

Using these values and applying the model formulas in Eqs. (1) and (2), respectively, we write

$$\theta_B = y'\big|_{x=L} = \theta_A + \frac{A_y}{2EI}L^2 + 0 - 0 + 0 - 0 - 0 + 0 + 0 + 0 - 0 = \frac{5M_0 L}{32EI}$$

$$y_B = y\big|_{x=L} = 0 + \theta_A L + \frac{A_y}{6EI}L^3 + 0 - 0 + 0 - 0 - 0 + 0 + 0 + 0 - 0 = -\frac{M_0 L^2}{32EI}$$

We report that

$$\boxed{\mathbf{A}_y = \frac{9M_0}{16L}\uparrow} \qquad \boxed{\theta_A = \frac{M_0 L}{8EI} ↻} \qquad \boxed{\theta_B = \frac{5M_0 L}{32EI} ↺} \qquad \boxed{y_B = \frac{M_0 L^2}{32EI}\downarrow}$$

● **Using MoS:** For applying the *method of superposition* and adapting to the formulas in Fig. 2, we may *turn* the original given beam in Fig. 8 about a vertical axis through 180° and *note* that it is equivalent to the superposition (*or* sum) of two differently loaded beams as shown in Fig. 10.

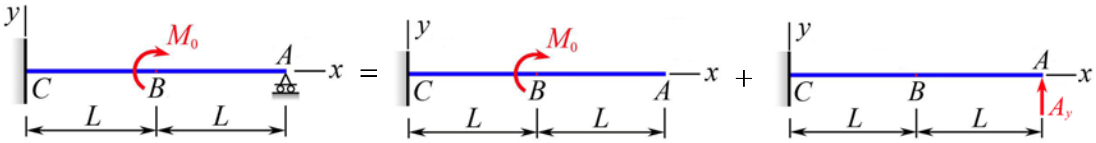


Fig. 10. Original given beam is equivalent to the superposition of two beams shown

Referring to the formulas for the 1st and 5th beams in Fig. 2, applying the **MoS**, and imposing the boundary condition of zero deflection at A, we write

$$y_A = -\left(\frac{M_0L^2}{2EI} + L\cdot\frac{M_0L}{EI}\right) + \frac{A_y(2L)^3}{3EI} = 0 \qquad \therefore\ A_y = \frac{9M_0}{16L}$$

$$\theta_A = \frac{M_0L}{EI} - \frac{A_y(2L)^2}{2EI} = -\frac{M_0L}{8EI}$$

$$\theta_B = \frac{M_0L}{EI} - \left\langle \frac{d}{dx}\left\{\frac{A_y x^2}{6EI}\left[3(2L)-x\right]\right\}\right\rangle\Bigg|_{x=L} = \frac{M_0L}{EI} - \frac{3L^2A_y}{2EI} = \frac{5M_0L}{32EI}$$

$$y_B = -\frac{M_0L^2}{2EI} + \left\{\frac{A_y x^2}{6EI}\left[3(2L)-x\right]\right\}\Bigg|_{x=L} = -\frac{M_0L^2}{2EI} + \frac{5A_yL^3}{6EI} = -\frac{M_0L^2}{32EI}$$

We report that

$$\boxed{\mathbf{A}_y = \frac{9M_0}{16L}\uparrow} \qquad \boxed{\theta_A = \frac{M_0L}{8EI}\circlearrowright} \qquad \boxed{\theta_B = \frac{5M_0L}{32EI}\circlearrowleft} \qquad \boxed{y_B = \frac{M_0L^2}{32EI}\downarrow}$$

Assessment of effectiveness. In this example, we see that the **MoMF** enables us to directly and successfully obtain the solutions, while the **MoS** requires some rotations and combinations of beams to eventually arrive at the same solutions. The **MoMF** is more straightforward than the **MoS**; but they are *about* equally effective in solving the problem in this example.

Example 4. A continuous beam AC with constant flexural rigidity EI and total length $2L$ has a roller support at A, a roller support at B, and a fixed support at C. This beam carries a linearly distributed load and is shown in Fig. 11. Determine (*a*) the vertical reaction force $\mathbf{A}_y$ and slope θ_A at A, (*b*) the vertical reaction force $\mathbf{B}_y$ and slope θ_B at B.

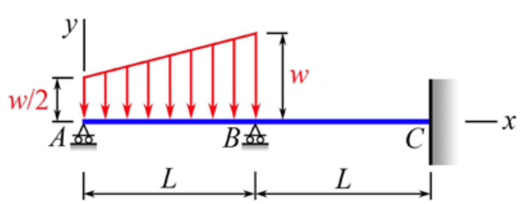


Fig. 11. Continuous beam AC carrying a linearly distributed load

Solution. The free-body diagram of the beam is shown in Fig. 12. We readily note that the beam is statically indeterminate to the *second* degree.

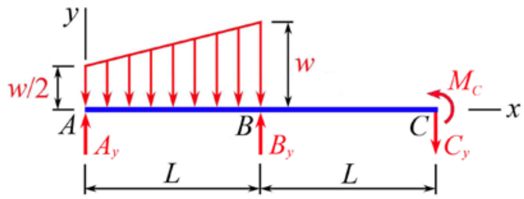


Fig. 12. Free-body diagram of the continuous beam *AC*

● **Using MoMF:** In applying the *method of model formulas* to this beam, we notice that the beam *AC* has a total length 2*L*, which will be the value for the *parameter L* in all model formulas in Eqs. (1) through (4). We see that the shear force V_A at left end *A* is equal to A_y, the moment M_A and deflection y_A at *A* are zero, the deflection y_B at *B* is zero, and the slope θ_C and deflection y_C at *C* are zero. Applying the model formulas in Eqs. (3) and (4) to the beam *AC* and using Eq. (2) to impose the condition that $y_B = y(L) = 0$ at *B*, in that *order*, we write

$$0 = \theta_A + \frac{A_y(2L)^2}{2EI} + 0 - \frac{-B_y}{2EI}(2L-L)^2 + 0 - \frac{w/2}{6EI}(2L)^3 - \frac{w-(w/2)}{24EIL}(2L)^4$$
$$+\frac{w}{6EI}(2L-L)^3 + \frac{w-(w/2)}{24EIL}(2L-L)^4 + 0 - 0$$

$$0 = 0 + \theta_A(2L) + \frac{A_y(2L)^3}{6EI} + 0 - \frac{-B_y}{6EI}(2L-L)^3 + 0 - \frac{w/2}{24EI}(2L)^4 - \frac{w-(w/2)}{120EIL}(2L)^5$$
$$+\frac{w}{24EI}(2L-L)^4 + \frac{w-(w/2)}{120EIL}(2L-L)^5 + 0 - 0$$

$$0 = 0 + \theta_A L + \frac{A_y}{6EI}L^3 + 0 - 0 + 0 - \frac{w/2}{24EI}L^4 - \frac{w-(w/2)}{120EIL}L^5 + 0 + 0 + 0 - 0$$

These *three* simultaneous equations yield

$$A_y = \frac{39wL}{140} \qquad \theta_A = -\frac{3wL^3}{140EI} \qquad B_y = \frac{31wL}{56}$$

Using these values and applying the model formula in Eq. (1), we write

$$\theta_B = y'\big|_{x=L} = \theta_A + \frac{A_y}{2EI}L^2 + 0 - 0 + 0 - \frac{w/2}{6EI}L^3 - \frac{w-(w/2)}{24EIL}L^4 + 0 + 0 + 0 - 0$$
$$= \frac{23wL^3}{1680EI}$$

We report that

$$\boxed{\mathbf{A}_y = \frac{39wL}{140}\uparrow} \qquad \boxed{\theta_A = \frac{3wL^3}{140EI}\circlearrowleft} \qquad \boxed{\mathbf{B}_y = \frac{31wL}{56}\uparrow} \qquad \boxed{\theta_B = \frac{23wL^3}{1680EI}\circlearrowleft}$$

● **Using MoS:** For applying the *method of superposition* and adapting to the formulas in Fig. 2, we may *turn* the original given beam in Fig. 10 about a vertical axis through 180° and *note* that it is equivalent to the superposition (*or* sum) of six differently loaded beams as shown in Fig. 13.

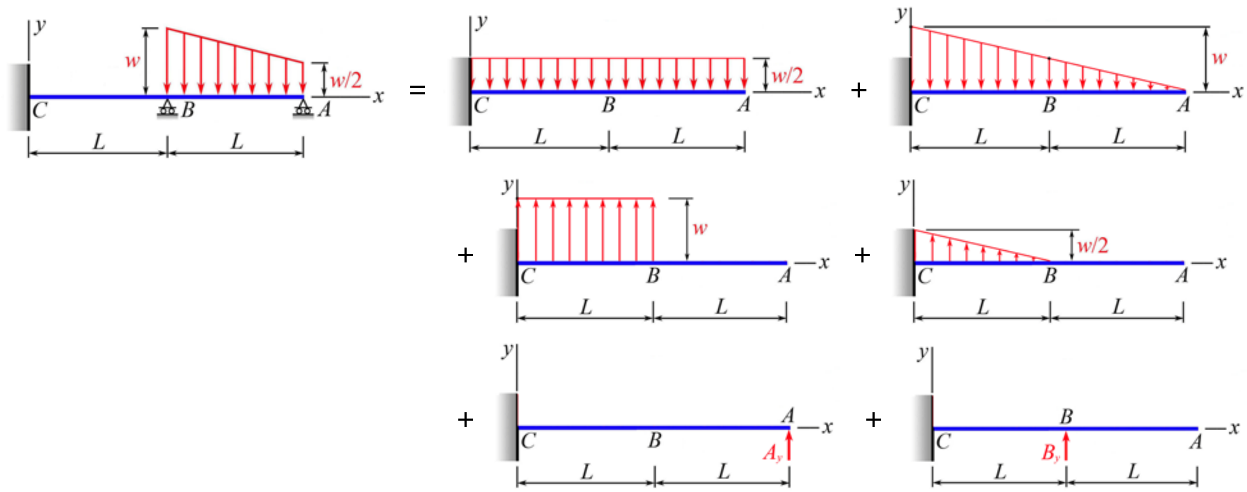


Fig. 13. Original given beam is equivalent to the superposition of six beams as shown

We have assumed that the reactions at A and B are $A_y \uparrow$ and $B_y \uparrow$, respectively. Referring to the formulas for the 1st, 2nd, 3rd, and 4th beams in Fig. 2 and applying the **MoS**, we impose the two boundary conditions $y_A = 0$ at A and $y_B = 0$ at B, successively, to write

$$-\frac{(w/2)(2L)^4}{8EI} - \frac{w(2L)^4}{30EI} + \left(\frac{wL^4}{8EI} + L\cdot\frac{wL^3}{6EI}\right) + \left[\frac{(w/2)\,L^4}{30EI} + L\cdot\frac{(w/2)\,L^3}{24EI}\right]$$
$$+\frac{A_y(2L)^3}{3EI} + \frac{B_yL^2}{6EI}\big[3(2L) - L\big] = 0$$

and

$$-\left\{\frac{(w/2)x^2}{24EI}\left[x^2+6(2L)^2-4(2L)\,x\right]\right\}\Bigg|_{x=L} - \left\{\frac{wx^2}{120(2L)EI}\left[10(2L)^3-10(2L)^2x+5(2L)\,x^2-x^3\right]\right\}\Bigg|_{x=L}$$
$$+\frac{wL^4}{8EI} + \frac{(w/2)\,L^4}{30EI} + \left\{\frac{A_y\,x^2}{6EI}\big[3(2L) - x\big]\right\}\Bigg|_{x=L} + \frac{B_yL^3}{3EI} = 0$$

The above two equations may be simplified and shown to be equivalent to the matrix equation

$$\begin{bmatrix} 640 & 200 \\ 10 & 4 \end{bmatrix}\begin{bmatrix} A_y \\ B_y \end{bmatrix} = \begin{bmatrix} 289wL \\ 5wL \end{bmatrix}$$

Solving the matrix equation, we obtain and report that

$$A_y = \frac{39wL}{140} \qquad \mathbf{A}_y = \frac{39wL}{140}\uparrow \qquad B_y = \frac{31wL}{56} \qquad \mathbf{B}_y = \frac{31wL}{56}\uparrow$$

Using the obtained values for A_y and B_y, referring to the formulas for the 1st, 2nd, 3rd, and 4th beams in Fig. 2, and applying the **MoS**, we write

$$\theta_A = \frac{(w/2)(2L)^3}{6EI} + \frac{w(2L)^3}{24EI} - \frac{wL^3}{6EI} - \frac{(w/2)\,L^3}{24EI} - \frac{A_y(2L)^2}{2EI} - \frac{B_yL^2}{2EI} = -\frac{3wL^3}{140EI}$$

$$
\begin{aligned}
\theta_B = &\left\langle \frac{d}{dx}\left\{ \frac{(w/2)\,x^2}{24EI}\left[x^2+6(2L)^2-4(2L)\,x\right]\right\}\right\rangle\Bigg|_{x=L} \\
&+\left\langle \frac{d}{dx}\left\{ \frac{w\,x^2}{120(2L)\,EI}\left[10(2L)^3-10(2L)^2\,x+5(2L)\,x^2-x^3\right]\right\}\right\rangle\Bigg|_{x=L} \\
&-\frac{wL^3}{6EI}-\frac{(w/2)\,L^3}{24EI}-\left\langle \frac{d}{dx}\left\{ \frac{A_y\,x^2}{6EI}\left[3(2L)-x\right]\right\}\right\rangle\Bigg|_{x=L}-\frac{B_yL^2}{2EI} \\
= &\frac{23wL^3}{1680EI}
\end{aligned}
$$

We report that

$$
\boxed{\theta_A = \frac{3wL^3}{140EI} \circlearrowright} \qquad \boxed{\theta_B = \frac{23wL^3}{1680EI} \circlearrowleft}
$$

Assessment of effectiveness. In this example, we see that the *method of model formulas* enables us to directly and successfully obtain the solutions, while the *method of superposition* requires some rotations and combinations of beams to eventually arrive at the same solutions. For the problem in this example, the road to the final solution is more straightforward when we use the **MoMF**, but it is more meandering and complex when we use the **MoS**. Clearly, the **MoMF** is more effective and has an edge over the **MoS** in this case.

III. Effective Teaching of the MoMF

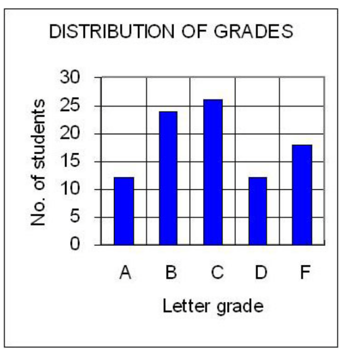
The *method of model formulas* is a general methodology that employs a set of *four equations* to serve as *model formulas* in solving problems involving statically indeterminate reactions, as well as slopes and deflections, of elastic beams. The first two model formulas are for the slope and deflection at any position x of the beam and contain rudimentary singularity functions, while the other two model formulas contain only traditional algebraic expressions. Generally, this method is more direct in solving beam deflection problems. Most students favor this method because they can solve problems in shorter time using this method and they score higher in tests.

The examples in Section II provide a variety of head-to-head comparisons between solutions by the *method of model formulas* and those by the traditional *method of superposition*. A **one-page excerpt** from the *method of model formulas*, such as that shown in Fig. 1, must be available to those who used this method. Past experience shows that the following steps form a pedagogy that can be used to effectively introduce and teach the *method of model formulas* to students to enrich their study and set of skills in finding statically indeterminate reactions and deflections of elastic beams in the undergraduate course of Mechanics of Materials:

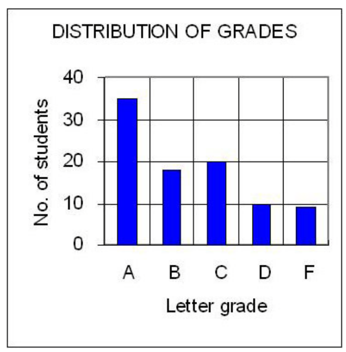
- First, teach the fundamental *method of integration* and the imposition of boundary conditions.
- Teach the rudiments of *singularity functions* and utilize them in the *method of integration*.
- Go over briefly the derivation[1] of the *four model formulas* in terms of *singularity functions*.

- Give students the heads-up on the following features in the *method of model formulas*:
  - No need to integrate or evaluate constants of integration.
  - Not prone to generate a large number of simultaneous equations **even if**
    - ▷ the beam carries multiple concentrated loads (forces or moments),
    - ▷ the beam has one or more simple supports *not* at its ends,
    - ▷ the beam has linearly distributed loads *not* starting at its left end,
    - ▷ the beam has linearly distributed loads *not* ending at its right end, and
    - ▷ the beam has non-uniform flexural rigidity but can be divided into uniform segments.
- Demonstrate solutions of several beam problems by the *method of model formulas*.
- Check the solutions obtained (e.g., comparing with solutions by another method).

In a 92-student Mechanics of Materials class (for sophomores and juniors) in the spring semester of 2010, the students were first taught the *method of integration* (**MoI**), then the *method of model formulas* (**MoMF**) in the study of deflection of beams. Based on *available* data: (*a*) when the students learn and use the **MoI** to solve statically indeterminate reactions and deflections of beams in a quiz, the results of their performance were 12 A's, 23 B's, 27 C's, 12 D's, and 18 F's, with overall class average equal to 69.1%; (*b*) when these same students learn and use the **MoMF** to solve statically indeterminate reactions and deflections of beams in a quiz, the results of their performance were 35 A's, 18 B's, 20 C's, 10 D's, and 9 F's, with overall class average equal to 79.7%. The grade distributions for these two quizzes are show in Fig. 14. The **MoMF** has appeared to be a more accessible method to the students in understanding the deflection of beams, and students favor this method in solving the problems.



(*a*) Quiz result on using **MoI**
(overall average = 69.1%)



(*b*) Quiz result on using **MoMF**
(overall average = 79.7%)

Fig. 14. Students' performance in using **MoMF**

IV. Concluding Remarks

In the *method of model formulas*, no explicit integration or differentiation is involved in applying any of the model formulas. The model formulas essentially serve to provide *material equations* (which involve and reflect the material property) besides the equations of static equilibrium of the beam that can readily be written. Selected model applied loads are illustrated in Fig. 1(*a*), which cover most of the loads encountered in undergraduate Mechanics of Materials. In the case of a nonlinearly distributed load on the beam, the model formulas may be modified by the user for such a load.

The *method of model formulas* is relatively new; it is best taught to students as an additional *or* alternative method after they have first learned the *method of integration*.[2-13] This new method is found to be more general and effective than the traditional *method of superposition*, as shown in Section II. Learning and using this new method will enrich students' study and set of skills in determining reactions and deflections of beams. Moreover, this new method provides engineers with a means to independently check their solutions obtained using traditional methods.

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