Teaching Deflections of Beams: Advantages of Method of Model Formulas versus Those of Conjugate Beam Method

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Objectives:

- Share the idea of teaching two methods:
  method of model formulas (MoMF), established in 2009,
  (I. C. Jong, "An Alternative Approach to Finding Beam Ractions and Deflections: MoMF," *IJEE*, pp. 1422-1427, 2009. Univ. of Arkansas)
  and *conjugate beam method* (CBM), established in 1921,
  (H. M. Westergaard, "Deflection of Beams by the Conjugate Beam Method," *JWSE*, pp.369-396, 1921. Univ. of Illinois).
- Provide comparisons between the new MoMF and the traditional CBM via head-to-head contrasting solutions of same problems.
- Enrich students' study and sets of skills in analyzing beam reactions and deflections.

An Effective Approach to Teaching the MoMF:

- First, teach the traditional *method of integration* (MoI), including the use of singularity functions.
- Go over briefly the derivation of the *four* model formulas in terms of singularity functions.
- Demonstrate solutions of several beam problems by the MoMF vs. the CBM and point out the advantages:
  - Needs no integration.
  - Handles well concentrated & distributed loads.
  - Uses just a page of excerpt of 4 model formulas.

Prerequisite to Using MoMF: Singularity Functions

$$< x-a >^n = (x-a)^n \quad \text{if} \quad x-a \geq 0 \quad \text{and} \quad n > 0$$

$$< x-a >^n = 1 \quad \text{if} \quad x-a \geq 0 \quad \text{and} \quad n = 0$$

$$< x-a >^n = 0 \quad \text{if} \quad x-a < 0 \quad \text{or} \quad n < 0$$

$$\int_{-\infty}^{x} < x-a >^n dx = < x-a >^{n+1} \quad \text{if} \quad n \leq 0$$

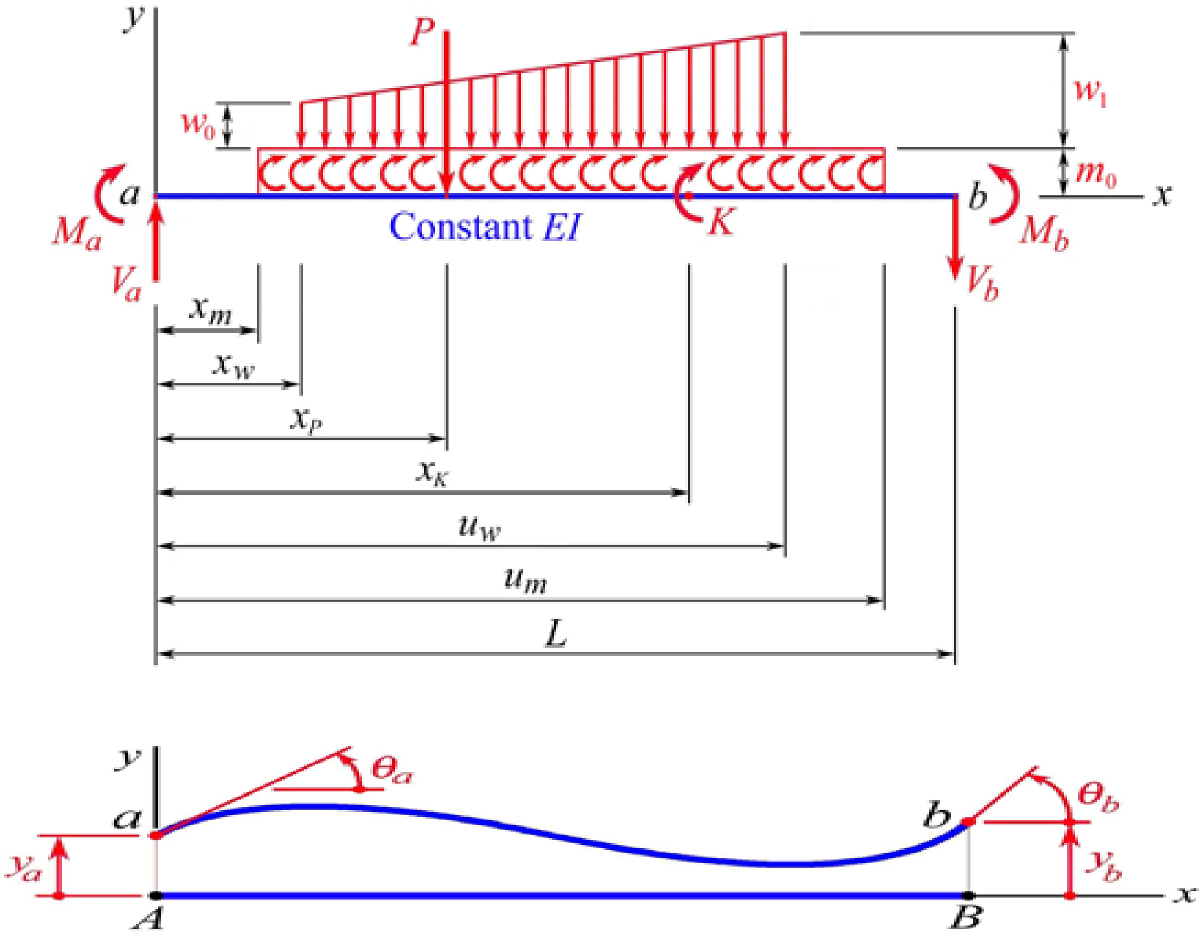
$$\int_{-\infty}^{x} < x-a >^n dx = \frac{1}{n+1} < x-a >^{n+1} \quad \text{if} \quad n > 0$$

$$\frac{d}{dx} < x-a >^n = n < x-a >^{n-1} \quad \text{if} \quad n > 0$$

$$\frac{d}{dx} < x-a >^n = < x-a >^{n-1} \quad \text{if} \quad n \leq 0$$

Excerpt from the **Method of Model Formulas:**

Model loads on beam *ab* & its positive deflections



Model Formulas: Eqs. (1) and (2)

$$y' = \theta_a + \frac{V_a}{2EI}x^2 + \frac{M_a}{EI}x - \frac{P}{2EI} < x - x_P >^2 + \frac{K}{EI} < x - x_K >^1 - \frac{w_0}{6EI} < x - x_w >^3$$
$$- \frac{w_1 - w_0}{24EI(u_w - x_w)} < x - x_w >^4 + \frac{w_1}{6EI} < x - u_w >^3 + \frac{w_1 - w_0}{24EI(u_w - x_w)} < x - u_w >^4$$
$$+ \frac{m_0}{2EI} < x - x_m >^2 - \frac{m_0}{2EI} < x - u_m >^2 \qquad (1)$$

$$y = y_a + \theta_a x + \frac{V_a}{6EI}x^3 + \frac{M_a}{2EI}x^2 - \frac{P}{6EI} < x - x_P >^3 + \frac{K}{2EI} < x - x_K >^2 - \frac{w_0}{24EI} < x - x_w >^4$$
$$- \frac{w_1 - w_0}{120EI(u_w - x_w)} < x - x_w >^5 + \frac{w_1}{24EI} < x - u_w >^4 + \frac{w_1 - w_0}{120EI(u_w - x_w)} < x - u_w >^5$$
$$+ \frac{m_0}{6EI} < x - x_m >^3 - \frac{m_0}{6EI} < x - u_m >^3 \qquad (2)$$

Model Formulas: Eqs. (3) and (4)

$$\theta_b = \theta_a + \frac{V_a L^2}{2EI} + \frac{M_a L}{EI} - \frac{P}{2EI}(L - x_P)^2 + \frac{K}{EI}(L - x_K) - \frac{w_0}{6EI}(L - x_w)^3 - \frac{w_1 - w_0}{24EI(u_w - x_w)}(L - x_w)^4 + \frac{w_1}{6EI}(L - u_w)^3 + \frac{w_1 - w_0}{24EI(u_w - x_w)}(L - u_w)^4 + \frac{m_0}{2EI}(L - x_m)^2 - \frac{m_0}{2EI}(L - u_m)^2 \quad (3)$$

$$y_b = y_a + \theta_a L + \frac{V_a L^3}{6EI} + \frac{M_a L^2}{2EI} - \frac{P}{6EI}(L - x_P)^3 + \frac{K}{2EI}(L - x_K)^2 - \frac{w_0}{24EI}(L - x_w)^4 - \frac{w_1 - w_0}{120EI(u_w - x_w)}(L - x_w)^5 + \frac{w_1}{24EI}(L - u_w)^4 + \frac{w_1 - w_0}{120EI(u_w - x_w)}(L - u_w)^5 + \frac{m_0}{6EI}(L - x_m)^3 - \frac{m_0}{6EI}(L - u_m)^3 \quad (4)$$

Guiding Rules in the CBM

Synthesized from: *Journal of the Western society of Engineers*, *Vol. XXVI, No. 11, pp.369-396*, 1921

Rule 1: Conjugate beam and given beam are of the same length.

Rule 2: Load on conjugate beam is M/EI of the given beam.

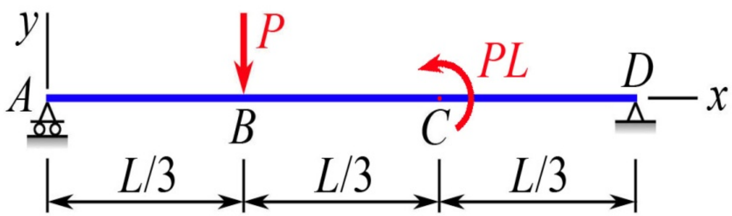
	Existing support condition in the **given beam**:	Corresponding support condition in the **conjugate beam**:
Rule 3:	Fixed end	Free end
Rule 4:	Free end	Fixed end
Rule 5:	Simple support at the end	Simple support at the end
Rule 6:	Simple support *not* at the end	Unsupported hinge
Rule 7:	Unsupported hinge	Simple support

Rule 8: Conjugate beam is in static equilibrium.

Rule 9: Slope of given beam is equal to the shear force in conjugate beam.

Rule 10: Deflection of given beam is equal to the moment in conjugate beam.

Example 1. Determine for the beam with EI (a) slopes θ_A and θ_D at A and D, (b) deflection y_B at B.



- Solution by MoMF:

Eq. (3): $$\theta_D = \theta_A + \frac{(5P/3)L^2}{2EI} - \frac{P}{2EI}\left(L - \frac{L}{3}\right)^2 + \frac{-PL}{EI}\left(L - \frac{2L}{3}\right)$$

Eq. (4): $$0 = \theta_A L + \frac{(5P/3)L^3}{6EI} - \frac{P}{6EI}\left(L - \frac{L}{3}\right)^3 + \frac{-PL}{2EI}\left(L - \frac{2L}{3}\right)^2$$

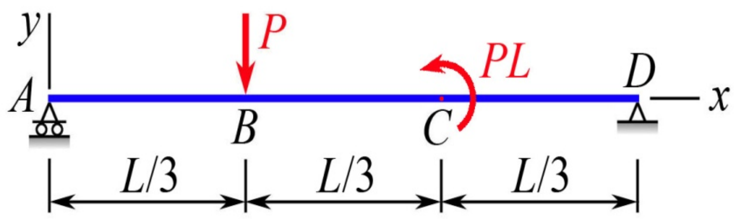
We obtain: $\theta_A = -\dfrac{14PL^2}{81EI}$ and $\theta_D = \dfrac{17PL^2}{162EI}$

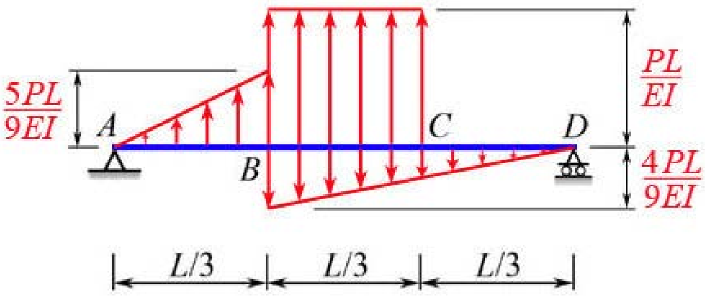
Eq. (2): $$y_B = y\big|_{x=L/3} = \theta_A\left(\frac{L}{3}\right) + \frac{5P/3}{6EI}\left(\frac{L}{3}\right)^3 = -\frac{23PL^3}{486EI}$$

We obtain: $y_B = \dfrac{23PL^3}{486EI} \downarrow$

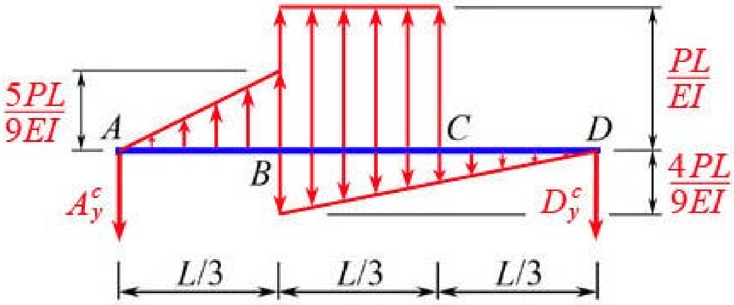
Example 1. (Continued)

- **Solution by CBM:**





Conjugate beam



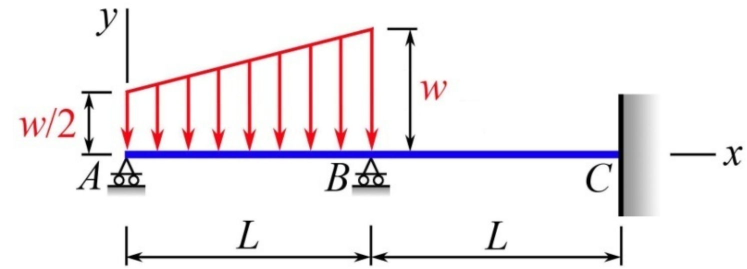
FBD of conjugate beam

Equilibrium of the conjugate beam yields: $A_y^c = \dfrac{14PL^2}{81EI}$ $D_y^c = \dfrac{17PL^2}{162EI}$

Applying the guiding rules in the CBM, we obtain:

$$\theta_A = \frac{14PL^2}{81EI} \circlearrowright \qquad \theta_D = \frac{17PL^2}{162EI} \circlearrowleft \qquad y_B = \frac{23PL^3}{486EI} \downarrow$$

Example 2. For the beam with EI, determine the reactions and slopes (*a*) $\mathbf{A}_y$ and θ_A at A, (*b*) $\mathbf{B}_y$ and θ_B at B.



■ Solution by MoMF: (assuming $\mathbf{A}_y = A_y\uparrow$ and $\mathbf{B}_y = B_y\uparrow$)

Eq. (3): $$0 = \theta_A + \frac{A_y(2L)^2}{2EI} - \frac{-B_y}{2EI}(2L-L)^2 - \frac{w/2}{6EI}(2L)^3 - \frac{w-(w/2)}{24EIL}(2L)^4 + \frac{w}{6EI}(2L-L)^3 + \frac{w-(w/2)}{24EIL}(2L-L)^4$$

Eq. (4): $$0 = \theta_A(2L) + \frac{A_y(2L)^3}{6EI} - \frac{-B_y}{6EI}(2L-L)^3 - \frac{w/2}{24EI}(2L)^4 - \frac{w-(w/2)}{120EIL}(2L)^5 + \frac{w}{24EI}(2L-L)^4 + \frac{w-(w/2)}{120EIL}(2L-L)^5$$

Eq. (2): $$0 = \theta_A L + \frac{A_y}{6EI}L^3 - \frac{w/2}{24EI}L^4 - \frac{w-(w/2)}{120EIL}L^5$$

Eq. (1): $$\theta_B = y'\big|_{x=L} = \theta_A + \frac{A_y}{2EI}L^2 - \frac{w/2}{6EI}L^3 - \frac{w-(w/2)}{24EIL}L^4$$

The preceding four equations yield the values:

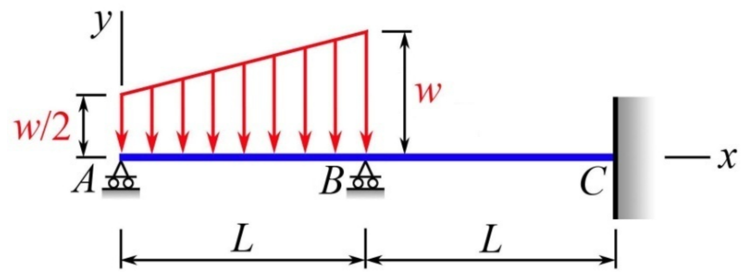
$$A_y = \frac{39wL}{140} \qquad \theta_A = -\frac{3wL^3}{140EI}$$

$$B_y = \frac{31wL}{56} \qquad \theta_B = \frac{23wL^3}{1680EI}$$

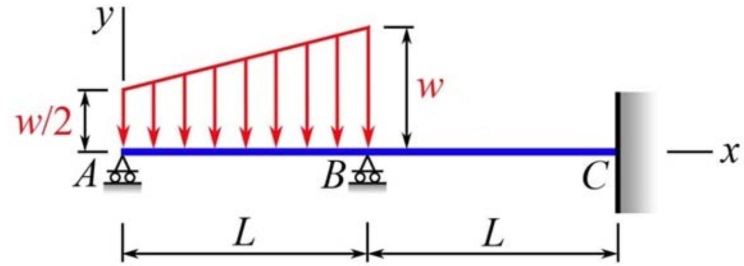
We report that the MoMF yields the following solutions:

$$\mathbf{A}_y = \frac{39wL}{140}\uparrow \qquad \theta_A = \frac{3wL^3}{140EI}\circlearrowright$$

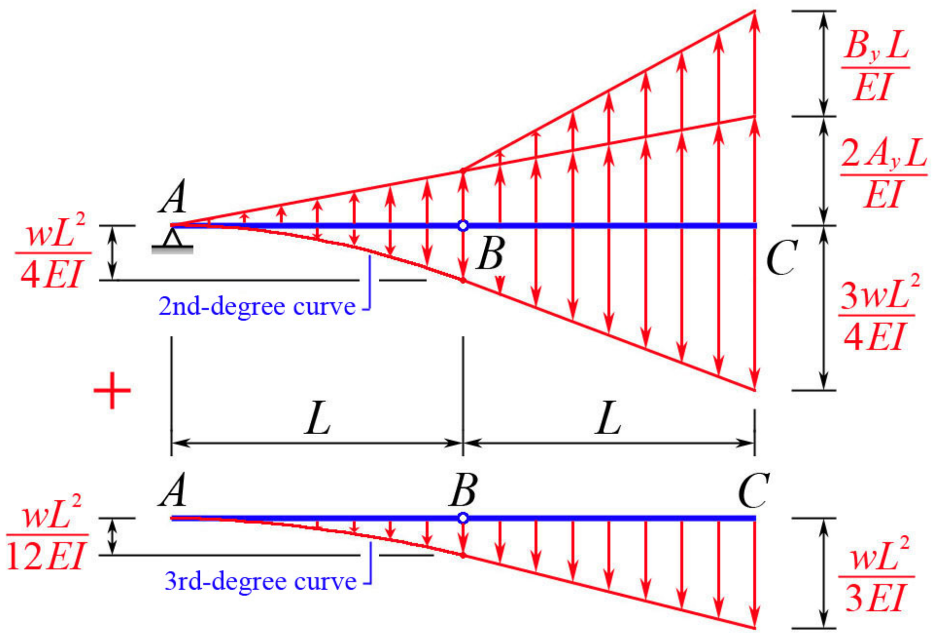
$$\mathbf{B}_y = \frac{31wL}{56}\uparrow \qquad \theta_B = \frac{23wL^3}{1680EI}\circlearrowleft$$



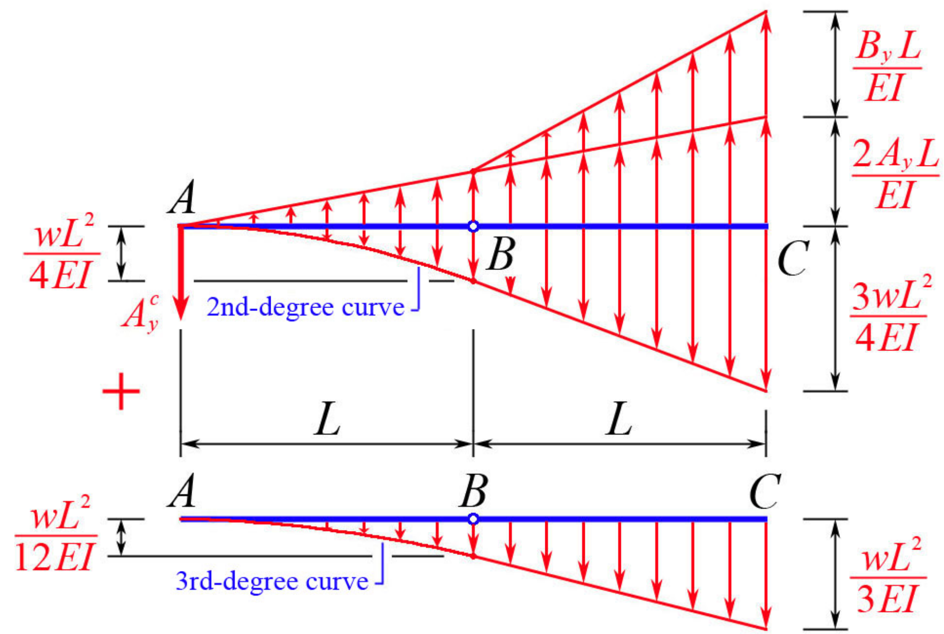
Example 2. (Continued)



■ Solution by the CBM:



Conjugate beam



FBD of conjugate beam

Equilibrium of the conjugate beam yields:

$$A_y = \frac{39wL}{140} \qquad B_y = \frac{31wL}{56} \qquad A_y^c = \frac{3wL^3}{140EI}$$

We report $\mathbf{A}_y = \frac{39wL}{140} \uparrow \qquad \mathbf{B}_y = \frac{31wL}{56} \uparrow$

Applying the guiding rules in the CBM, we have

$$\theta_A = V_A^c = -A_y^c = -\frac{3wL^3}{140EI} \qquad \theta_B = V_B^c = \frac{23wL^3}{1680EI}$$

We report $\theta_A = \frac{3wL^3}{140EI} \circlearrowright \qquad \theta_B = \frac{23wL^3}{1680EI} \circlearrowleft$

Conclusions

- The *method of model formulas* requires the use of just a printed page of excerpt of model loads and four model formulas.
- The four model formulas serve to provide material equations, besides the equations of equilibrium for a beam.
- The formulas in the MoMF can account for *most* of the loads encountered in undergraduate Mechanics of Materials.
- The *conjugate beam method* require the use of the first 7 guiding rules to construct the conjugate beam and draw the free-body diagram of the conjugate beam. Then, apply the last 3 guiding rules to compute and report the requested solutions.
- Generally, the MoMF is a more *direct* and *effective* method than the CBM in determining reactions and deflections of beams.

THANK YOU

Questions?