Guiding Rules in the **Conjugate Beam Method**

(**Excerpt** from an *IJEE* Paper)

Courtesy: *Int. J. Engng. Ed.*, Vol. 26, No. 6, pp. 1422-1427, 2010

icjong@uark.edu

1.1 Pedagogy of the conjugate beam method. The conjugate beam method is actually a natural extension of the *moment-area theorems*. It is an elegant, efficient, and powerful method published by Westergaard [1] some nine decades ago, although some considered Mohr (1868) and Breslau (1865) to have prior influences. Elementary presentation of this method did appear in early textbooks in mechanics of materials [2, 3]. For reasons unknown, this method is missing in most such current textbooks. The pedagogy of the conjugate beam method lies in teaching and applying the rules in this method [1, 11]. These rules are summarized as follows:

Rule 1: The conjugate beam and the given beam are of the same *length*.

Rule 2: The load on the conjugate beam is the *elastic weight*, which is the bending moment M in the given beam divided by the flexural rigidity EI of the given beam.
(This *elastic weight* is taken to act upward if the *bending moment* is positive — to cause top fiber in compression — in beam convention.)

For each *existing support condition* of the given beam, there is a *corresponding support condition* for the conjugate beam. The correspondence is given by rules 3 through 7 as follows:

	Existing support condition in the **given beam:**	Corresponding support condition in the **conjugate beam:**
Rule 3:	Fixed end	Free end
Rule 4:	Free end	Fixed end
Rule 5:	Simple support at the end	Simple support at the end
Rule 6:	Simple support *not* at the end	Unsupported hinge
Rule 7:	Unsupported hinge	Simple support

Rule 8: The conjugate beam is in static *equilibrium*.

Rule 9: The *slope* of the given beam at any cross section is given by the "*shear force*" at that cross section of the conjugate beam.
(This *slope* is positive, or counterclockwise, if the "*shear force*" is positive — tending to rotate the beam element clockwise — in beam convention.)

Rule 10: The *deflection* of the given beam at any point is given by the "*bending moment*" at that point of the conjugate beam.
(This *deflection* is upward if the "*bending moment*" is positive — tending to cause the top fiber in compression — in beam convention.)

1.2 Illustration of the pedagogy. A combined beam, with a constant flexural rigidity EI, fixed supports at its ends A and D, a hinge connection at B, and carrying a concentrated force $\mathbf{P}$ at C, is shown in Fig. 2. Determine *(a)* the vertical reaction force $\mathbf{A}_y$ and the reaction moment $\mathbf{M}_A$ at A, *(b)* the deflection y_B of the hinge at B, *(c)* the slopes θ_{BL} and θ_{BR} just to the left and just to the right of the hinge at B, respectively, and *(d)* the slope θ_C and the deflection y_C at C.

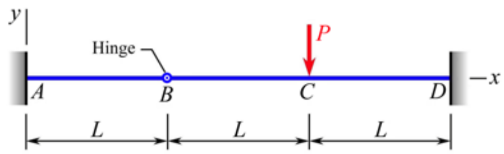

Fig. 2. Statically indeterminate beam with a hinge connection.

Solution. We note that the beam given in Fig. 2 is statically indeterminate to the *first* degree. Using $\mathbf{A}_y$ as the redundant unknown, we may assume that the reaction force and reaction moment at A are as shown in Fig. 3. Drawing the moment-diagram *by parts*, we may construct the corresponding conjugate beam as shown in Fig. 4.

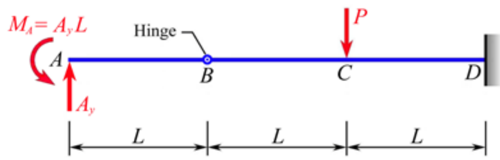

Fig. 3. Reactions at end A of the beam in Fig. 2.

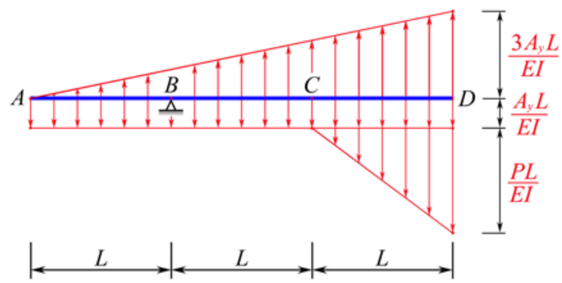

Fig. 4. Conjugate beam for the beam in Figs. 2 and 3.

Notice in Figs. 2 and 4 the following key points:

- The conjugate beam and the given beam are of the same length.
- The fixed ends at A and D in the given beam change to free ends at A and D in the conjugate beam.
- The unsupported hinge at B in the given beam changes to a simple support at B in the conjugate beam.
- The *elastic weight* acting on the conjugate beam comes from the bending moment M in the given beam divided by the flexural rigidity EI of the given beam.

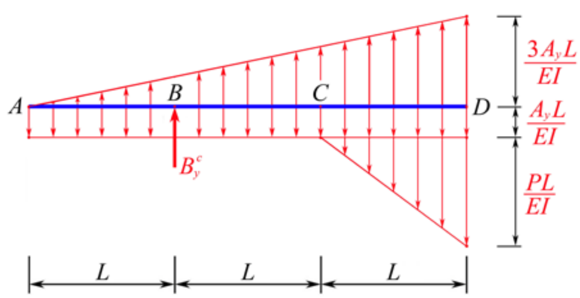

Fig. 5. Free-body diagram of the conjugate beam in Fig. 4.

The conjugate beam in Fig. 4 and the free body of the conjugate beam in Fig. 5 are in static *equilibrium*. Referring to the entire free-body diagram in Fig. 5, we write

$+\circlearrowleft \Sigma M_B^c = 0:$

$$L \cdot \frac{3L}{2} \cdot \frac{3A_yL}{EI} - \frac{L}{2} \cdot 3L \cdot \frac{A_yL}{EI} - \left(L + \frac{2L}{3}\right) \cdot \frac{L}{2} \cdot \frac{PL}{EI} = 0 \tag{1}$$

This equation yields

$$A_y = \frac{5P}{18} \qquad M_A = A_yL = \frac{5PL}{18} \tag{2}$$

We report that

$$\boxed{\mathbf{A}_y = \frac{5P}{18} \uparrow} \qquad \boxed{\mathbf{M}_A = \frac{5PL}{18} \circlearrowleft}$$

Referring to Fig. 5, we write

$+\uparrow \Sigma F_y^c = 0:$

$$B_y^c + \frac{3L}{2} \cdot \frac{3A_yL}{EI} - 3L \cdot \frac{A_yL}{EI} - \frac{L}{2} \cdot \frac{PL}{EI} = 0 \tag{3}$$

This equation yields

$$B_y^c = \frac{PL^2}{12EI} \tag{4}$$

Using the above obtained values, referring to Fig. 5, and applying the rules in the *conjugate beam method* in Section 1.1, we may compute and report the requested quantities as follows:

$$y_B = M_B^c = \frac{L}{3} \cdot \frac{L}{2} \cdot \frac{A_yL}{EI} - \frac{L}{2} \cdot \frac{A_yL^2}{EI} = -\frac{5PL^3}{54EI} \qquad \boxed{y_B = \frac{5PL^3}{54EI} \downarrow}$$

$$\theta_{BL} = V_{BL}^c = \frac{L}{2} \cdot \frac{A_yL}{EI} - \frac{A_yL^2}{EI} = -\frac{5PL^2}{36EI} \qquad \boxed{\theta_{BL} = \frac{5PL^2}{36EI} \circlearrowright}$$

$$\theta_{BR} = V_{BR}^c = V_{BL}^c + B_y^c = -\frac{2PL^2}{36EI} \qquad \boxed{\theta_{BR} = \frac{2PL^2}{36EI} \circlearrowright}$$

$$\theta_C = V_C^c = \frac{2L}{2} \cdot \frac{2A_yL}{EI} + B_y^c - 2L \cdot \frac{A_yL}{EI} = \frac{PL^2}{12EI} \qquad \boxed{\theta_C = \frac{PL^2}{12EI} \circlearrowleft}$$

$$y_C = M_C^c = \frac{2L}{3} \cdot \frac{2L}{2} \cdot \frac{2A_yL}{EI} + LB_y^c - L \cdot \frac{2A_yL^2}{EI} = -\frac{11PL^3}{108EI} \qquad \boxed{y_C = \frac{11PL^3}{108EI} \downarrow}$$

Based on the preceding solutions, deflections of the beam in Fig. 2 are depicted in Fig. 6.

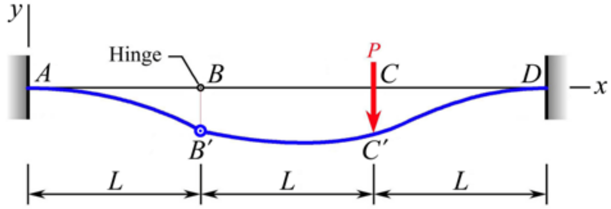

Fig. 6. Obtained configuration of deflections of the beam in Fig. 2.

4. Conclusions

Conventional wisdom in the solution of a differential equation governing the deflection of a beam, *or* the behavior of a certain physical system, expects and requires that adequate boundary conditions be available and be satisfied before a unique solution can be obtained. Westergaard's conjugate beam method employs *support conditions* and hence by-passes the protocol requiring adequate *boundary conditions* for solving problems of beam deflections. This approach works well because boundary conditions have, in fact, been taken into account in the conjugate beam method when the support conditions are specified in the beginning stages of the solutions.

More support conditions than boundary conditions are usually known for beams in neutral equilibrium. The conjugate beam method can readily handle five basic support conditions: fixed end, free end, simple support at the end, simple support *not* at the end, and unsupported hinge. This method usually requires no explicit integration in the solution, and it requires good skills in statics in the operation. The conjugate beam method is suitable for learning by sophomores and juniors; and it has been taught, tested, and highlighted in the course MEEG 3013 Mechanics of Materials at the University of Arkansas for several years. In the analysis of beam deflections, this method is the one method most frequently preferred by the students.

The conjugate beam method is unique and outstanding. It is the only analytical method that can be applied to investigate the deflection of a beam in neutral equilibrium. This method is concise and efficacious. Above all, it attests that some early ideas in engineering could be still useful today and one should pay attention to them.

References

1. H. M. Westergaard, "Deflections of Beams by the Conjugate Beam Method," *Journal of the Western Society of Engineers*, Vol. XXVI, No. 11, 1921, pp. 369-396.
2. S. Timoshenko and G. H. MacCullough, *Elements of Strength of Materials* (3rd Edition), Van Nostrand, New York, NY, 1949.
3. F. L. Singer and A. Pytel, *Strength of Materials* (4th Edition), Harper & Row, New York, NY, 1987.
4. S. H. Crandall, C. D. Norman, and T. J. Lardner, *An Introduction to the Mechanics of Solids* (2nd Edition), McGraw-Hill, New York, NY, 1972.
5. R. J. Roark and W. C. Young, *Formulas for Stress and Strain* (5th Edition), McGraw-Hill, New York, NY, 1975.
6. A. Pytel and J. Kiusalaas, *Mechanics of Materials*, Brooks/Cole, Pacific Grove, CA, 2003.
7. J. M. Gere, *Mechanics of Materials* (6th Edition), Brooks/Cole, Pacific Grove, CA, 2004.
8. R. G. Budynas and J. K. Nisbett, *Shigley's Mechanical Engineering Design* (8th Edition), McGraw-Hill, New York, NY, 2008.
9. F. P. Beer, E. R. Johnston, Jr., J. T. DeWolf, and D. F. Mazureck, *Mechanics of Materials* (5th Edition), McGraw-Hill, New York, NY, 2009.
10. I. C. Jong, "An Alternative Approach to Finding Beam Reactions and Deflections: Method of Model Formulas," *International Journal of Engineering Education*, Vol. 25, No. 1, 2009, pp. 65-74.
11. I. C. Jong, "Effective Teaching and Learning of the Conjugate Beam Method: Synthesized Guiding Rules," *Proceedings of the 2004 ASEE Annual Conference & Exposition*, Salt Lake City, UT, 2004.