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TECHNICAL PAPERS

Deflections of Beams by the Conjugate Beam Method.

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I. INTRODUCTION.

1. *An Introductory Discussion of the Standard Methods of Finding Deflections of Beams.*—Three methods are commonly used in determining deflections of beams: The first is the purely analytical "double integration" method. The second is the graphical "funicular polygon" method or "string polygon" method. The third is the semi-geometrical "moment area" method. Each of these three methods has a particular field where it is most useful. By the analytical double integration method the equation of the elastic curve is determined; namely, by integration of an "equation of flexure" of the form $d^2y/dx^2 = \pm M/EI$. If the equation of the elastic curve (the deflected center-line) is the result called for, then it is natural, at least in a number of cases, to apply this method, which leads to fairly simple solutions when the load is of a simple nature. On the other hand, when the load is more complex, then the method would in most cases require a rather elaborate determination of integration constants, and this feature is undesirable. For example, six concentrated loads on a simple beam, dividing it into seven sectors, would require the determination of fourteen integration constants. The analytical integration method was used as early as 1744 by Euler,** and is important because, both logically and historically, it is the fundamental method from which the others have been derived.

The second method, that of the string polygon, was developed by O. Mohr† who showed in 1868 that the deflected curve may be found as a string curve or string polygon. This graphical method is useful when the load is complex, consisting, for example, of a large number of concentrated forces. A disadvantage of the method is that it gives results only in one specific case at the time. Moreover, the method is limited to the degree of approximation which can be obtained by graphical construction.

The third method, the moment area method, has appeared in more than one form, and it has proved itself useful in a large variety of both simple and complicated cases. It makes it possible to determine the deflection at one definite point without first finding the equation of the whole elastic curve. At the same time the method is well adapted for the purpose of determining formulas for the deflection at any point, that is, of determining the equation of the elastic curve. The moment

†O. Mohr, Beitrag zur Theorie der Holz und Eisenkonstruktionen, Zeitschrift des Architekten- und Ingenieurvereines zu Hannover, 1868.

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**Euler, Methodus inveniendi lineas curvas, etc., Lausanne, 1744, see Additamentum "De curvis elasticis."

area method was deduced in its first form by O. Mohr (1868)†. When proving the string polygon method, he recognized that when both the bending moments and the deflections can be found by means of string polygons, then it would also be possible to determine deflections as if they were bending moments. Mohr showed how the moment area principle derived in this way could be used with advantage in finding the deflections of simple beams, and afterward he applied the results in an analysis of continuous beams. At about the same time, or not much later, C. E. Greene*, at the University of Michigan, discovered the moment area principle in a distinctly different form. Greene's principle determines the deflections of a cantilever, or in general, the deflections of any beam, measured from any tangent to the elastic curve. These deflections are found to be proportional to moments of areas in the moment diagram. The basis of the proof is that the double integrals which express deflections can be interpreted as proportional to such moments of moment areas. Evidently, Mohr's and Greene's moment area principles are different. Each has its field where it is most useful: for example, Mohr's principle is preferable in the case of simple beams, Greene's in the case of cantilevers.

Later Mueller-Breslau** extended Mohr's original moment area principle in such a way that it became directly applicable to beams with any type of supports, and also to trusses. The extended method, applied to beams, includes as special cases both Mohr's and Greene's original principles. One of its main features is the use of an additional beam in which the bending moments are equal to or proportional to the deflections of the given beam. This beam, introduced by Mueller-Breslau, will here be called the "conjugate beam," and, accordingly, we shall call the extended moment area method the " conjugate beam method."

The pages which follow will be devoted to a discussion of the "conjugate beam method," its derivation, and its use in finding deflections of statically determinate beams and in the general analysis of statically indeterminate beams. In the treatment of statically indeterminate beams a method of selecting the conjugate beams will be used which departs slightly from customary methods. In other respects it is the plan to follow the usual way of presenting the subject.

The conjugate beam method requires an apparatus of investigation which is, of course, slightly more complicated than, for example, that of Greene's original principle. But when once established the operation of the extended method is in any case as simple as that of Greene's principle, and, in addition, the extended method has the advantage of a much wider range of direct applicability than the original more limited principles. This will be shown on the pages which follow.

2. *Definitions and Notation.*—We shall speak about the *given beam* and about the *conjugate beam*. The *given beam* is the beam of which the deflections are to be determined. The *conjugate beam* is a fictitious beam which corresponds to the given beam, and which is introduced for the purpose of analysis. It has the same length as the given beam. It is defined as a beam which is supported and loaded in such a way that its moment diagram becomes identical with the diagram of the deflections of the given beam. Or, by definition, the deflections of the given beam can be found as bending moments in the conjugate beam. Points on the two beams having the same distance, say, from the left end, are considered as "corresponding" or as "the same."

†See the paper just quoted. See also O. Mohr, Abhandlungen aus dem Gebiete der technischen Mechanik, 2 ed., 1914, pp. 342-374.

*According to J. E. Boyd, Strength of Materials, ed. 1917, p. 153, Greene began teaching the moment area method in 1873.

**See H. Mueller-Breslau, Beitrag zur Theorie des Fachwerks, Zeitschrift des Architekten und Ingenieurvereines zu Hannover, v. 31, 1885, p. 418; also his "Graphische Statik," v. II., 1, ed. 1892, pp. 99-120.

The following notation will be used:

w=distributed load on the given beam, per unit length; w is considered positive downwards.

V=vertical shear in the given beam, equal to the sum of vertical forces to the left of the section considered, with these forces considered positive upwards.

M=bending moment in the given beam; M is considered positive when causing compression at the top, tension at the bottom.

w', V', and M'=load per unit length, shear,, and bending moment, respectively, in the conjugate beam; they are considered positive in the same directions as w, V, and M.

θ=the slope of the elastic curve (that is, of the deflected center line), of the given beam, θ is considered positive clockwise.

y=deflection of the given beam, considered positive downwards.

x=horizontal distance, positive toward the right.

L=length of span.

EI=modulus of elasticity times moment of inertia of cross-section.

3. *The Six Diagrams Characterizing the Action of a Beam.*—Fig. 1 shows a simple beam which carries a distributed load w given by the diagram shown at top.

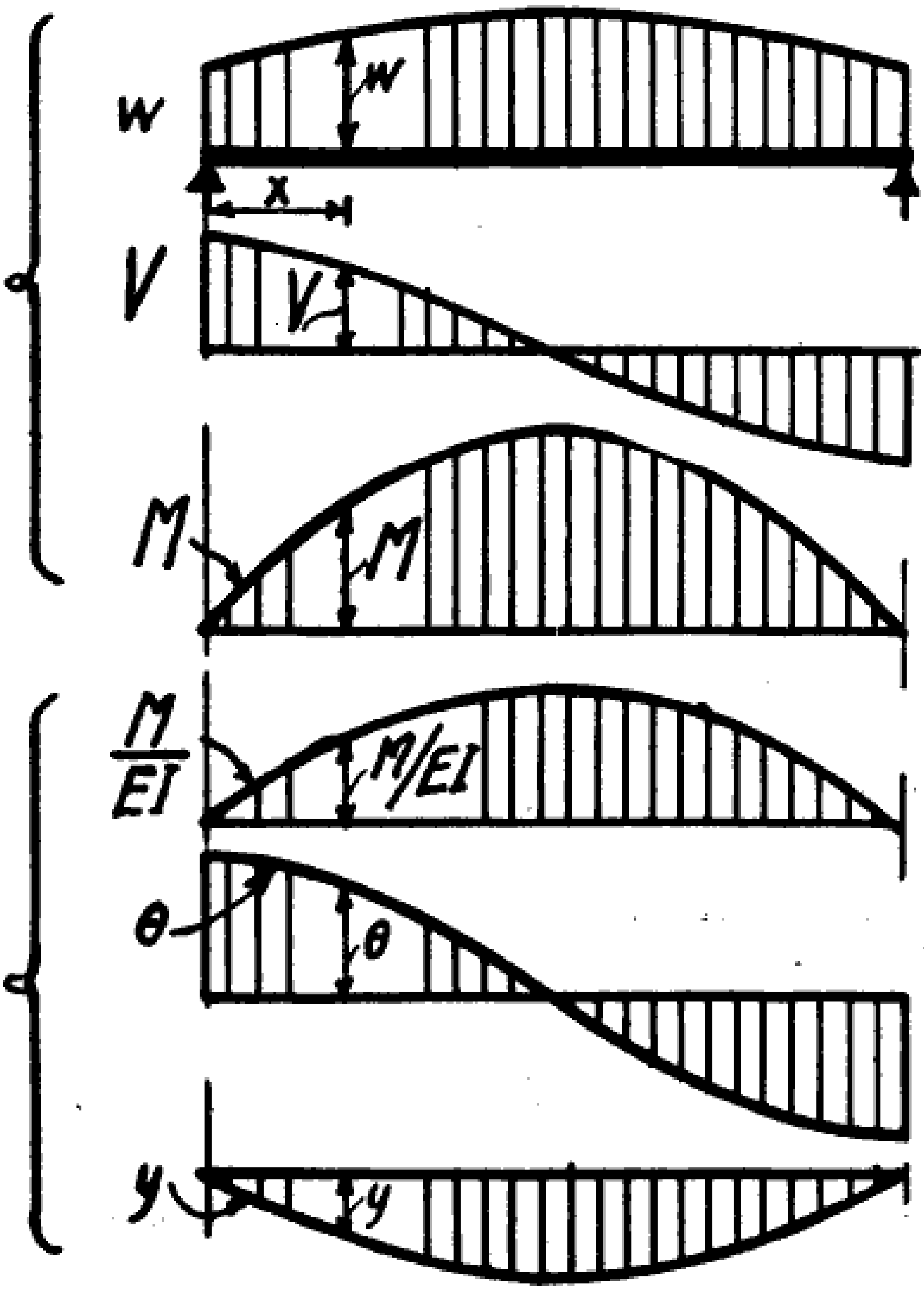

Fig. 1.

The action of the beam may be characterized by the six diagrams shown in Fig. 1. They are placed above one another, with common abscissas x. We shall consider these diagrams in two separate groups with three in each:

In the first group the ordinates are:

w (=load), V (=shear), and M (=bending moment), respectively.

In the second group the ordinates are:

M/EI, Θ (= slope), and y (= deflection), respectively.

Also in the case of any other type of beam,—cantilever, fixed, continuous, etc.,—the action of the beam may be described by means of six diagrams of this kind. In any case the following relations will exist between the diagrams:

In the first group:

$$V=\frac{dM}{dx}\;;\qquad -w=\frac{dV}{dx}=\frac{d^2M}{dx^2} \tag{1}$$

In the second group:

$$\theta=\frac{dy}{dx}\;;\qquad -\frac{M}{EI}=\frac{d\theta}{dx}=\frac{d^2y}{dx^2} \tag{2}$$

The relation $\Theta = dy/dx$ is simply a statement that the slope is the derivative of the deflection. The other relations occur in the usual theory of flexure. The negative sign of $-M/EI$ in equation (2) is in accordance with the choice of the positive direction of the deflections y (positive downwards).

The relations (1) and (2) between the diagrams may be interpreted graphically or geometrically as follows:

(1) The shear is the slope in the moment diagram. Minus the load per unit length is the slope in the shear diagram. (2) The ordinate Θ is the slope in the y-diagram. Minus the ordinate M/EI is the slope in the Θ-diagram.

The factor EI determines the relation between the diagrams of the first group and the diagrams of the second group. The factor EI may be a constant, or it may vary from point to point. Usually we shall assume EI=constant, in which case the ordinates in the M/EI-diagram are proportional to the ordinates of the M-diagram. Then these two diagrams will be similar, or, with some particular choice of scales, they will be identical.

4. *The Moment Area Principle and the Load Diagram of the Conjugate Beam.*—By our definition of the conjugate beam its bending moments M' are to be equal to the deflections y of the given beam, that is,

$$M' = y \tag{3}$$

By a proper loading of a beam it will always be possible to make its bending moment diagram take any given shape. The problem is then to find the particular load diagram w', and some particular method of support, which would cause the moments in the conjugate beam to be equal to the deflections of the given beam, such as stated in equation (3). We shall first determine the load diagram, w'. Equations (1) serve this purpose. As they apply to any beam, we can make them apply to the conjugate beam by substituting the valves w', V', and M' for w, V, and M. That is, we have:

$$V'=\frac{dM'}{dx}\;;\qquad -w'=\frac{dV'}{dx}=\frac{d^2M'}{dx^2} \tag{4}$$

Substituting $M' = y$ in accordance with (3) and comparing with (2) we find the solution:

$$w' = M/EI \tag{5}$$
$$V' = \theta \tag{6}$$

These two relations combined with (3),

$$M' = y \tag{3}$$

express the moment area principle, which may now be stated as follows: *The load diagram of the distributed loads acting on the conjugate beam is the same as the M/EI-diagram of the given beam. The slopes of the given beam are equal to the shears of the conjugate beam. The deflections of the given beam are equal to the bending moments of the conjugate beam.* The "moment areas" in question are, strictly speaking, not moment areas, but areas of the M/EI-diagram. It is these areas which are acting as loads on the fictitious "conjugate beam."

It remains for us to determine the character of the supports of the conjugate beam.

5. *The Supports and Other Special Points of the Conjugate Beam.*—Equation (5) gives complete information as to the forces acting on the conjugate beam on any stretch within which there is continuity in the deflections and slopes of the given beam. On the other hand, at special points, such as at the ends, or where the given beam has a hinge, there may be, and will be, in general, a discontinuity in the application of equations (2) and (4). Consequently, at such points there may be forces acting, other than the distributed forces given by equation (5). There may be concentrated forces or couples. These forces or couples must counterbalance the distributed load of the M/EI-diagram, and might therefore be produced as reactions supplied by the supports, provided these supports are placed properly. The supports of the conjugate beam may be different from those of the given beam. We shall indicate rules by which the character of the conjugate beam, its supports, hinges, etc., may be determined. In some cases there may be more than one solution, but in such cases we shall choose the simplest possible type of conjugate beam.

The following rules may be indicated: At points where the deflections or the slopes of the given beam have definite given values, the moments and the shears of the conjugate beam must be made to assume those definite values. And especially, *at points where the deflection of the given beam is zero, the moment in the conjugate beam must be made zero, and where the slope of the given beam is zero, the shear in the conjugate beam must be made zero.* This law leads to five specific rules for the selection of supports of the conjugate beam, and for the placing of its hinges, free ends, and other special points. Fig. 2 will illustrate the five cases, which are num-

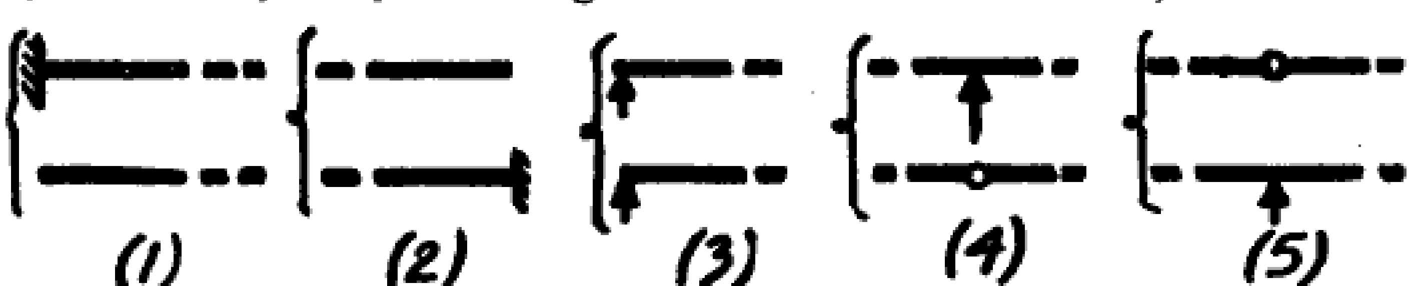

Fig. 2.

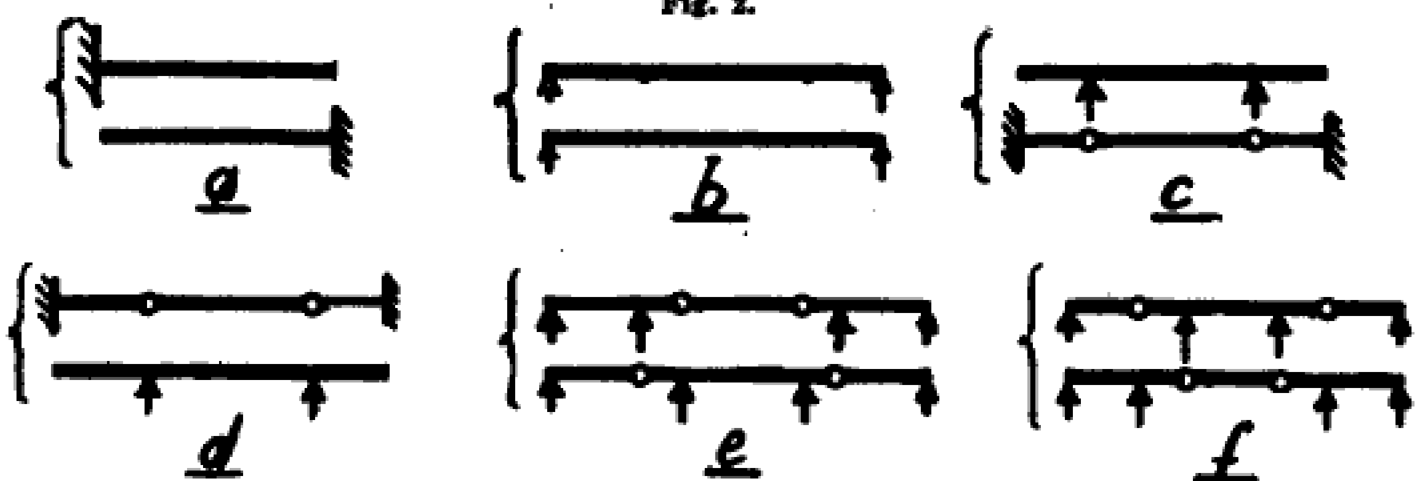

Fig. 1.

Upper Line: Given Beam. Lower Line: Conjugate Beam.

bered in the figure in the order in which they will be discussed; in each case a part of the given beam is shown above in the figure, while the corresponding part of the conjugate beam is shown directly below:

(1) A rigidly fixed end of the given beam (case (1), Fig. 2) is a point where its deflection and slope are zero. At such a point the conjugate beam must have zero bending moment and zero shear. This condition is established by letting the conjugate beam have a free unsupported end at that point. We might, instead of that, place a simple support at the point and then stipulate that the reaction at the point must be zero; but the former solution appears to be the simpler one and will be given the preference.

(2) A free end of the given beam, for example, of a cantilever, is a point where both the deflection and the slope may be different from zero (case (2), Fig. 2). Correspondingly, it must be arranged that the bending moment and the shear in the conjugate beam at that point may become different from zero. A fixed-end support of the conjugate beam at that point is the simplest arrangement which will cause an end moment and an end shear with the values which are necessary for maintaining equilibrium.

(3) A simple support at the end of the given beam (case (3), Fig. 2) causes the deflection at that point to be zero, but allows the slope to become different from zero. That is, in the conjugate beam the conditions at that point should cause the moment to be zero, but should provide for a shear which may be different from zero. The simplest way of establishing such conditions is by letting the conjugate beam end at the point and there have a simple support. The reaction from that simple support is equal to the shear in the conjugate beam, and must therefore be equal to the slope of the given beam.

(4) Next, consider a simple support not at the ends (case (4), Fig. 2). The given beam is assumed to be continuous over that support. The deflection at the point is zero. The slope may be different from zero, but must have the same value immediately to the left and to the right of the point. Therefore, in the conjugate beam the moment at the point must be zero, and the shear may have a value other than zero, but must have the same value immediately to the left and to the right of the point. An unsupported hinge in the conjugate beam is the simplest arrangement by which this condition can be established.

(5) A hinge in the original beam (see case (5), Fig. 2) is a point where the two adjoining parts must have the same deflection but may have different slopes. Hence the conjugate beam has the same moment immediately to the right and to the left of the point, but may have different shears. This condition is established when the conjugate beam has a simple support at the point, furnishing a single-force reaction only.

It is seen that Fig. 2 represents completely the five rules which have just been indicated.

6. *The Conjugate Beam in Some Definite Cases.*—The five rules illustrated in Fig. 2 will now be applied in a number of definite cases. Fig. 3 shows the application to the important types of statically determinate beams. In each case the given beam is shown above, the conjugate beam right below. In the case of the cantilever in Fig. 3*a*, rule (1) applies to the left end, rule (2) to the right end. Hence, the conjugate beam corresponding to a cantilever is a cantilever fixed at the opposite end. We verify the result that the conjugate beam must be fixed or held at the right end, by noting that the beam must be free at the left end, and that it must be in equilibrium.

Fig. 3*b* shows a simple beam. Rule (3) applies to both ends. The result is that the conjugate beam is a simple beam, like the given beam. That the concentrated forces acting at the ends of the conjugate beam can be found as reactions from simple supports, follows from the fact that the conjugate beam must be in equilibrium.

Other cases of statically determinate beams are shown in Fig. 3*c, d, e,* and *f*. The application of the rules of Fig. 2 in each separate case is easily recognized. It should be noted that in all these cases of statically determinate beams the conjugate beams are statically determinate. Because of this property it becomes unnecessary to indicate the particular elastic properties of the various pieces of the conjugate beam. It is enough to state that the individual pieces of the conjugate beams may be considered as rigid bodies.

A certain reciprocity may be noted in Fig. 3; if the conjugate beam in any of the cases were made the given beam, then the original given beam would become the conjugate beam. In other words, in all these cases the given beam is the conjugate beam of the conjugate beam. This reciprocity is found already in Fig. 2 where rules (1) and (2) are reciprocal, likewise (4) and (5), while (3) may be considered as its own reciprocal.

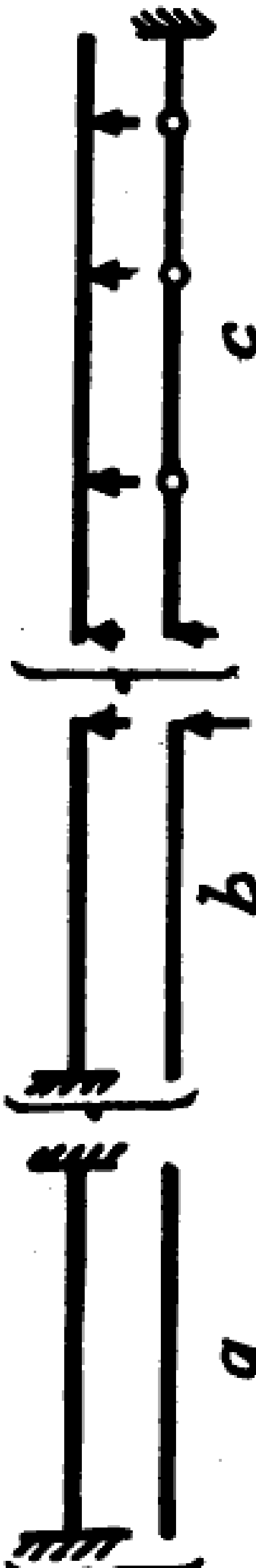

Fig. 4.

Fig. 4 shows three typical cases of statically indeterminate beams; that is, beams which have too many supports to allow the determination of the reactions and of the shear and moment diagrams by means of the ordinary statical conditions alone, without taking the elastic deformations into account. In Fig. 4 the given beams are shown above. The corresponding conjugate beams, derived in each case by the rules of Fig. 2 are shown below. The given beam in Fig. 4*a* is fixed at both ends. An application of rule (1) to both ends gives a conjugate beam which is free at both ends.

This conjugate beam is "incompletely supported" in the sense that unless the load diagram has a special character, the beam could not be in equilibrium. That is, the load diagram, in this case the M/EI-diagram, must be adjusted, like the buoyancy forces on a floating body, in order to establish equilibrium. It will be shown later that this process of adjustment of the M/EI-diagram supplies the remainder of the conditions which are needed in determining the unknown reactions. Or, the conditions of equilibrium of the conjugate beam furnish a means of determining not only the deflections of the given statically indeterminate beam, but also its moments, shears, and reactions. Similar remarks may be made about the conjugate beams in Fig. 4*b* and *c*, which are also "incompletely supported." In Fig. 4*b* the given beam is fixed at the left end, simply supported at the right end. An application of rule (1) to the left end, rule (3) to the right end, leads to a conjugate beam which is simply supported at the right end only. Equilibrium is secured by a particular adjustment of the M/EI-diagram. In Fig. 4*c* the given beam is continuous over three spans and has an overhanging end to the right. Rules (3), (4), and (2) apply at the special points. Instead of using the unsupported hinges in Fig. 4*c* one might place supports under them and then stipulate that the reactions from those supports must be zero. However, we prefer here to indicate as "the conjugate beam" that which is shown in Fig. 4*c* and which has unsupported hinges.

7. *Deflections Measured from Other Lines than from the Original Undeflected Center Line.*—Assume that it is desired to measure the deflections of the simple beam *AB* in Fig. 5 from the chord *CD,* where *C* and *D* are points of the beam, deflecting with the beam. The fundamental equations (2) apply to the deflections from *CD*. Furthermore, as far as the deflections from *CD* are concerned, the beam may be considered as simply supported on *CD* at points *C* and *D*. The actual reactions at *A* and *B* may then be considered as external forces acting at the free ends of the beam (any reaction may be considered as an applied external load). The corresponding conjugate beam is shown below in Fig. 5. The load acting on the conjugate beam is the *M/EI*-diagram of the original beam, supported at *A* and *B*.

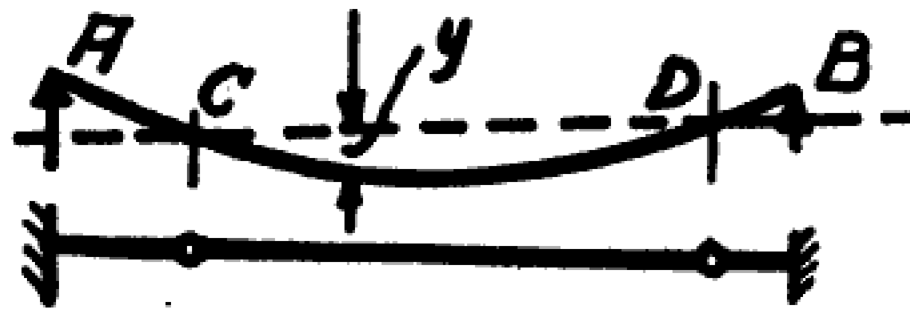

Fig. 5.

A similar example is illustrated in Fig. 6, where the deflections *y* are measured from the tangent at *C*. After the actual bending moment at the fixed end *A* has been determined, this bending moment is considered as an external couple. Then the beam may be analyzed as if supported at *C*. The corresponding conjugate beam consists of two cantilevers, and is shown below. In this way the deflections of any beam from any of its tangents may be found. That is, Greene's principle, determining the deflections from the tangent, is covered as a special case by this particular application of the principle of the conjugate beam.

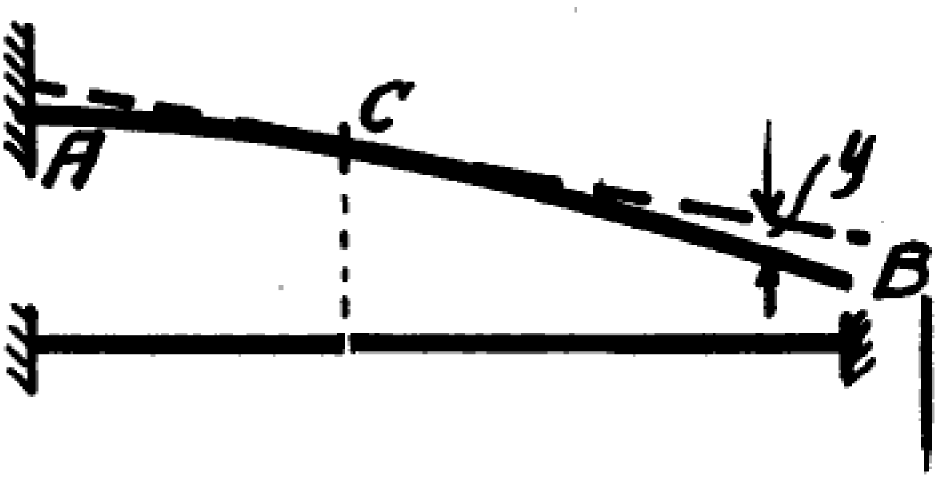

Fig. 6.

II. STATICALLY DETERMINATE BEAMS.

8. *General Remarks.*—We are now ready to apply the conjugate beam method to a number of definite cases. We begin with a study of deflections of statically determinate beams. In the specific cases treated it will be assumed that unless otherwise stated, the stiffness factor *EI* is a constant throughout the length of the beam. Also, unless otherwise stated, the deflections and slopes will be measured relative to the undeflected center line of the beam.

9. *Cantilevers.*—A cantilever of length *L,* fixed at the left end, will be considered. The conjugate beam was shown to be a cantilever fixed at the opposite end,—such as indicated in Fig. 3*a*.

A. *Concentrated load at the end.*—The M/EI-diagram is a triangle, as shown in Fig. 7. The bending moments are negative. That is, the M/EI-load is a load upward on the conjugate beam. The conjugate beam, which is fixed at the right end, is shown in Fig. 7, with the M/EI-load acting on it. Since that load acts upwards, it gives positive moments and shears in the conjugate beam. This result agrees with the fact that the slopes and deflections of the given beam must be positive (with the previously indicated notation, the deflections are positive downward, and the slopes are beam occur at the free end. The slope of the given beam at the free end is equal to the shear at the fixed end of the conjugate beam, or equal to minus the area of the M/EI-diagram, or

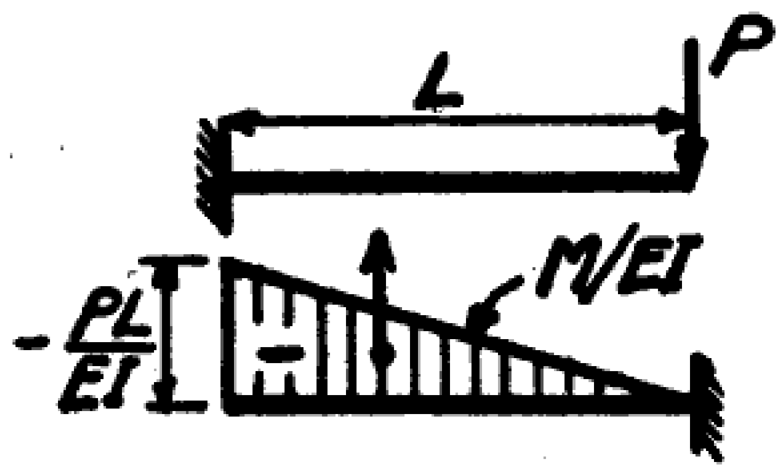

Fig. 7.

$$\theta_{max} = \frac{1}{2}\frac{PL^2}{EI} \tag{7}$$

The deflection at the right end of the original beam is the moment at the fixed end of the conjugate beam. The resultant of the M/EI-load passes through the centroid of the M/EI-diagram; that is, it has a moment arm equal to $(2/3)L$ with respect to the right end. The M/EI-load acts upwards, giving positive moments in the conjugate beam; or, positive deflections in the given beam. The maximum moment in the conjugate beam, equal to the right end deflection in the given beam, is then

$$y_{max} = \left(\frac{1}{2}\frac{PL^2}{EI}\right)\left(\frac{2L}{3}\right) = \frac{PL^3}{3EI} \tag{8}$$

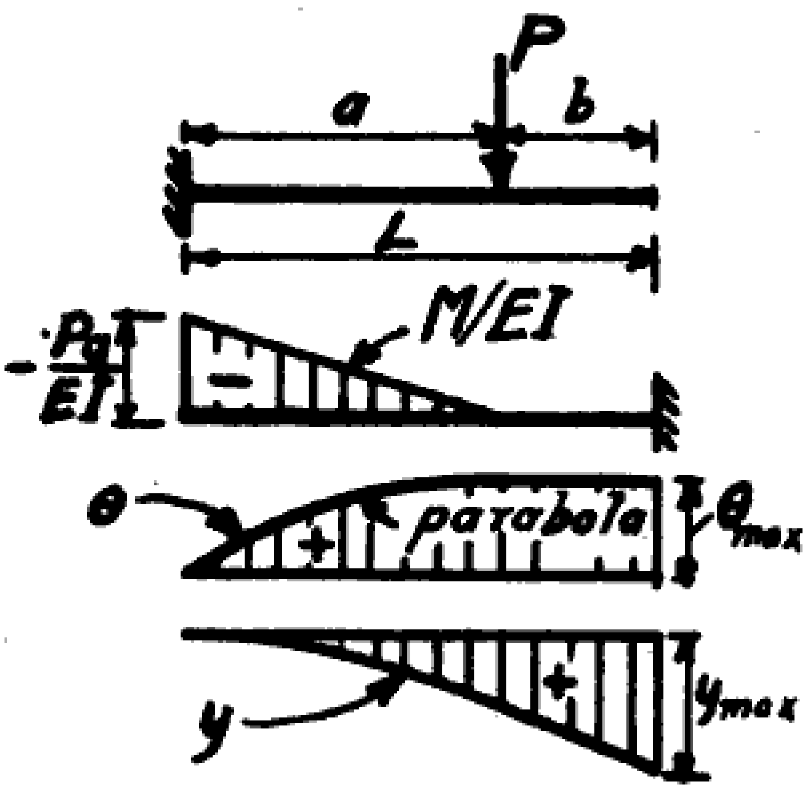

Fig. 8.

Slopes and deflections at other points may be found in a similar manner, and thus the complete slope and deflection diagrams may be obtained.

B. *Cantilever loaded by a concentrated force P at any point.*—The notation for distances is given in Fig. 8, which shows the given beam at top, and the conjugate beam, loaded with the M/EI-diagram right below. The maximum slope, as in the preceding case, is equal to minus the area of the M/EI diagram, that is,

$$\theta_{max} = +\frac{1}{2}\frac{Pa^2}{EI} \tag{9}$$

The moment at the fixed end of the conjugate beam, that is, the deflection at the free end of the given beam, is

$$y_{max} = +\frac{1}{2}\,\frac{Pa^2}{EI}\left(\frac{2}{3}a+b\right) \tag{10}$$

The shapes of the complete Θ and y-diagrams are shown below in Fig. 8.

C. *Cantilever with load uniformly distributed.*—Fig. 9 shows the given beam above, and the conjugate beam, loaded with the M/EI-diagram, below. The parabolic area of the M/EI-diagram has an average ordinate equal to one-third of the greatest ordinate. Its centroid is at a distance $(3/4)L$ from the right end. Hence we find for the right end:

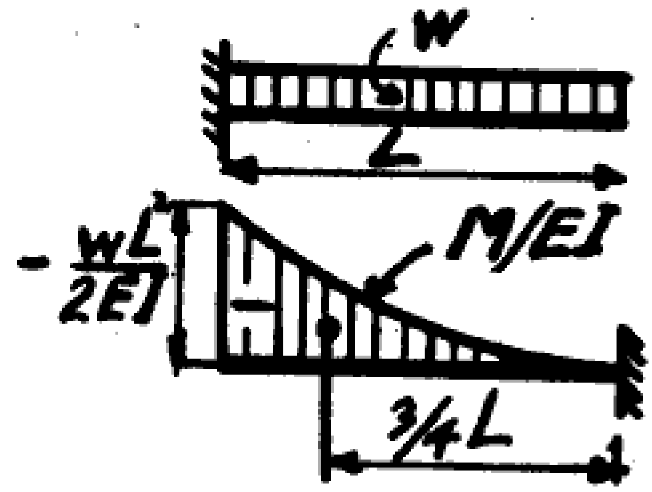

Fig. 9.

$$\theta_{max} = \frac{1}{3}\left(\frac{wL^2}{2EI}\right)\cdot L = \quad \frac{wL^3}{6EI} \tag{11}$$

and

$$y_{max} = \left(\frac{wL^3}{6EI}\right)\cdot\frac{3}{4}\,L = \quad \frac{wL^4}{8EI} \tag{12}$$

10. *Simple Beams.*—A simple beam with span L will be considered. The conjugate beam was shown to be a simple beam,—see Fig. 3*b*. This is the particular case in which the conjugate beam is the same as the given beam. Mohr's original theory was applied to that case. The action under various types of load will now be investigated.

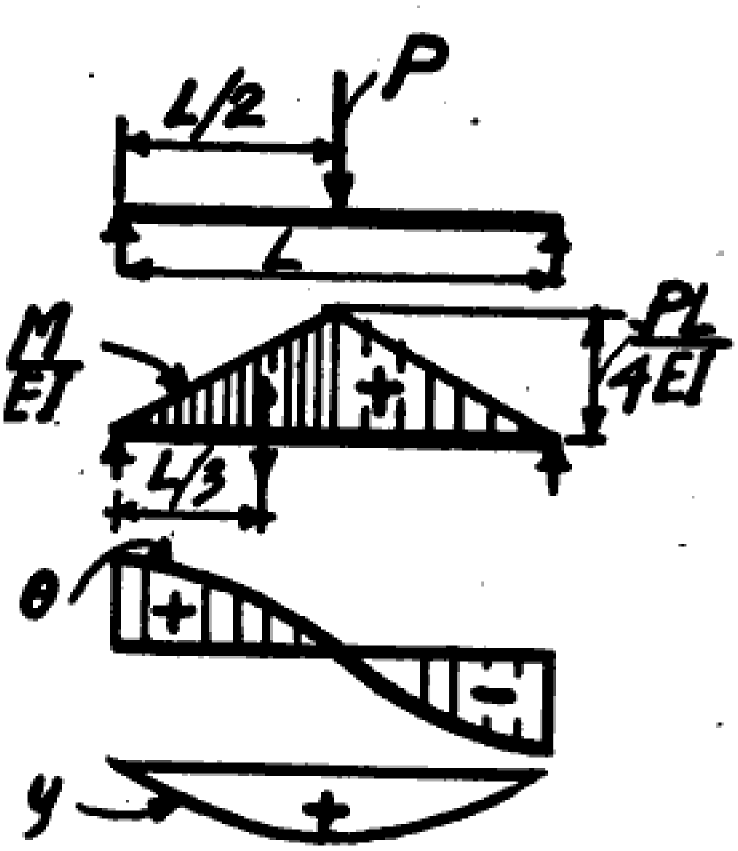

Fig. 10.

A. *Simple beam, concentrated load P at center.*—The M/EI-diagram is a triangle. It is shown in Fig. 10 as a load acting on the conjugate beam. The maximum slope in the given beam occurs at the left end. It is equal to the left end shear in the conjugate beam. This left end shear is equal to the reaction, which, on account of the symmetry, is one-half the area of the M/EI-diagram. Hence the following expression for the maximum slope:

$$\theta_{max} = \frac{1}{2} \cdot \frac{L}{2} \cdot \frac{PL}{4EI} = \frac{PL^2}{16EI} \tag{13}$$

The centroid of the left half of the M/EI-diagram is one-third of the span from the left end; hence the maximum moment in the conjugate beam, or the maximum deflection of the given beam, is expressed as

$$y_{max} = \frac{PL^2}{16EI} \cdot \frac{L}{3} = \frac{PL^3}{48EI} \tag{14}$$

B. *Simple beam, two equal concentrated loads, symmetrically placed.*—Fig. 11 gives the notation and shows the trapezoidal M/EI-diagram, which acts as a load on the conjugate beam. The slope at the left end and the deflection at the center are determined as the left end reaction and as the moment at the center of the conjugate beam. The values are:

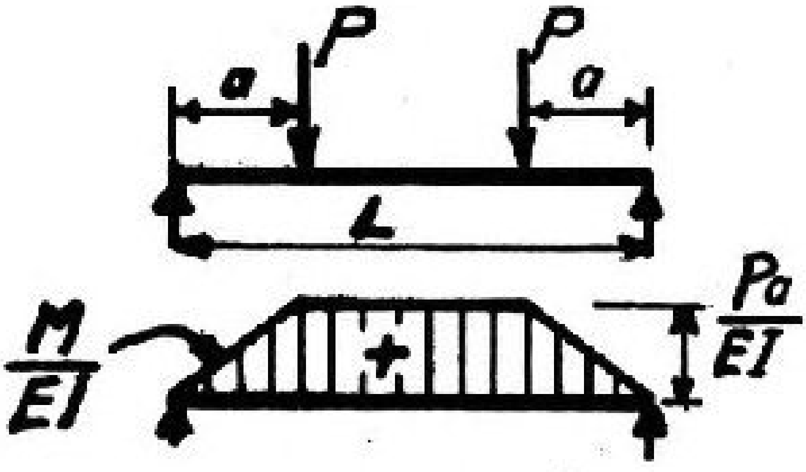

Fig. 11.

$$\theta_{max} = \frac{Pa\,(L-a)}{2EI}, \qquad y_{max} = \frac{Pa}{24EI}\left(3L^2 - 4a^2\right) \tag{15}$$

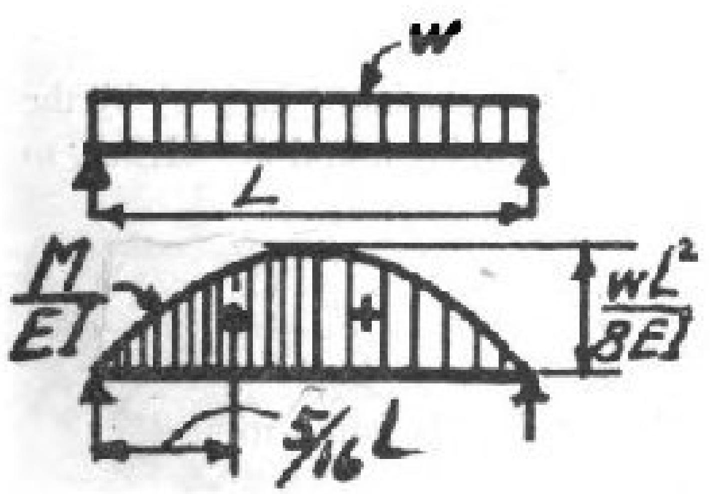

Fig. 12.

C. *Simple beam uniformly loaded* (Fig. 12).—The moment diagram is a parabola with the maximum ordinate $M = (1/8)wL^2$. The end slope is one-half the area of the M/EI-diagram, that is,

$$\theta_{max} = \frac{2}{3} \cdot \frac{wL^2}{8EI} \cdot \frac{L}{2} = \frac{wL^3}{24EI} \tag{16}$$

The centroid of the left half of the parabolic area is $(5/16)L$ from the left end, that is, the moment in the conjugate beam at the center, or, the maximum deflection of the given beam, is

$$y_{max} = \frac{wL^3}{24EI} \cdot \frac{5L}{16} = \frac{5\,wL^4}{384EI} \tag{17}$$

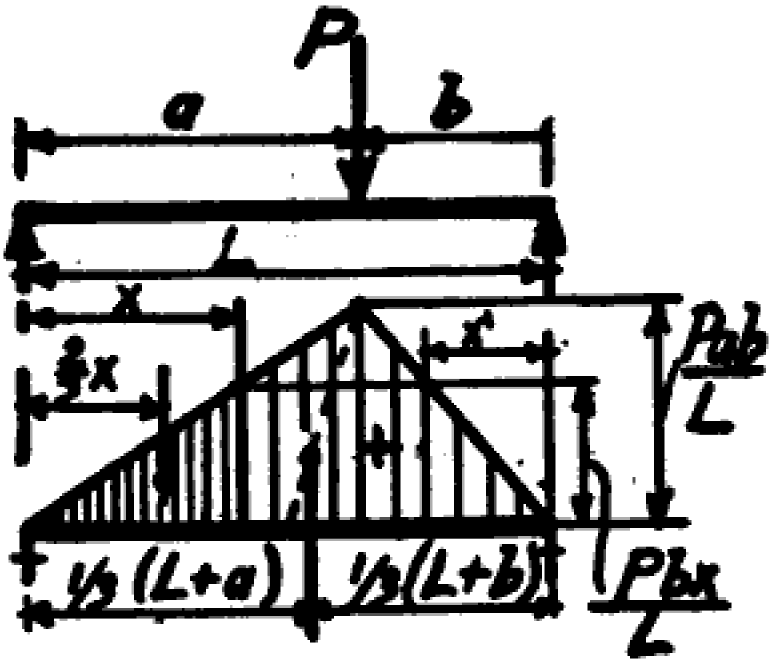

Fig. 13.

D. *Simple beam, concentrated load at any point.*—Fig. 13 gives the notation. The M/EI-diagram is a triangle with altitude $Pab/(EIL)$. The horizontal distance of the centroid of the triangle, measured from the right end, is $(1/3)\ (L+b)$; hence, the left end shear in the conjugate beam, or the left end slope in the given beam, is

$$\theta_A = \frac{L}{2}\cdot\frac{Pab}{EIL}\cdot\frac{L+b}{3L} = \frac{Pab\ (L+b)}{6EIL} \tag{18}$$

The moment in the conjugate beam or the deflection of the given beam at the distance x from the left end, when x is assumed less than a, is then

$$y = \theta_A \cdot x - \left(\frac{Pbx}{EIL}\cdot\frac{x}{2}\right)\cdot\frac{x}{3}$$

or

$$y = \frac{Pbx}{6EIL}\left(a(L+b) - x^2\right) \tag{19}$$

To the right of the load the deflection may be found by replacing x in (19) by the distance x' measured from the right end, and by interchanging a and b. Hence, to the right of the load we have the deflection

$$y' = \frac{Pax'}{6EIL}\left(b\ (L+a) - x'^2\right) \tag{20}$$

By substituting $x = a$ in (19) the deflection under the load is found to be

$$y_P = \frac{Pa^2b^2}{3EIL} \tag{21}$$

When P is in the right half of the beam then the location of the maximum deflection is found by differentiation of (19). Thereby the distance from the left end to the point of maximum deflection is found to be

$$x = \sqrt{(1/3)a(L+b)} \tag{22}$$

The point of maximum deflection could also be found, perhaps more directly, as the point of maximum moment in the conjugate beam, that is, as the point of zero shear in the conjugate beam.

E. *Simple beam loaded by couples at the ends of the span* (Fig. 14).—The slopes at the ends of the span have a particular interest. The moments M_A and M_B, which are applied at the ends as shown in Fig. 14, cause a trapezoidal M/EI-diagram. This M/EI-trapezoid may be considered as consisting of the two triangles which are marked *I* and *II*, and which have their centroids over the third-points of the span. Triangle *I* acting as a load on the conjugate beam is carried two-thirds by the left support and one-third by the right support; triangle *II*, in the same way, one-third by the left support, two-thirds by the right support. The shears in the conjugate beam, or the end slopes of the given beam, are then:

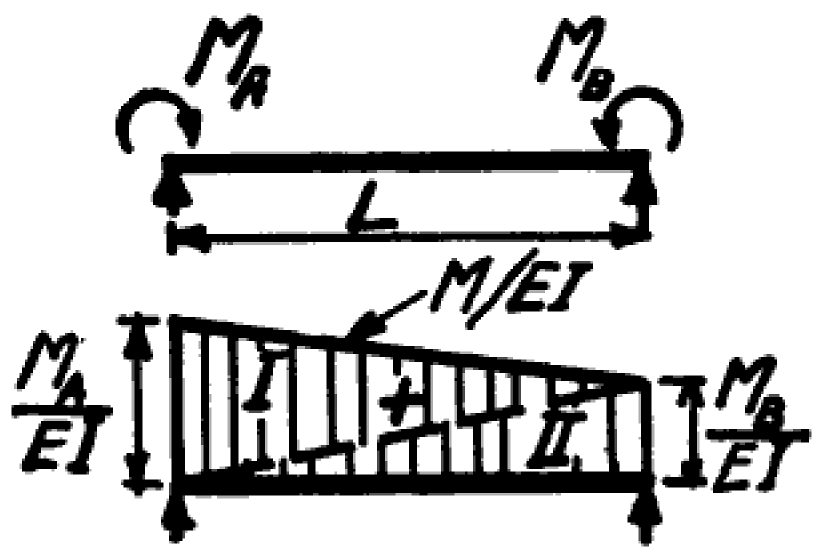

Fig. 14.

At the left end:

$$\theta_A = \frac{L}{6EI}\left(2M_A + M_B\right) \tag{23}$$

At the right end:

$$\theta_B = -\frac{L}{6EI}\left(M_A + 2M_B\right) \tag{24}$$

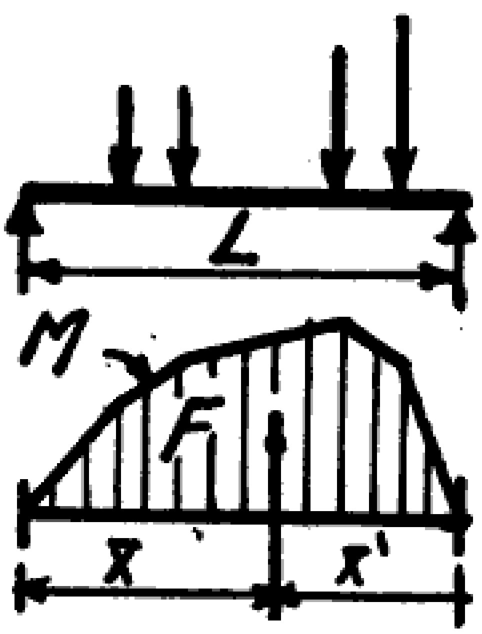

Fig. 15.

F. *Simple beam, any moment diagram.*—Let F be the area of the moment diagram (Fig. 15). Let the distance of its centroid from the left end be $\bar{x}$, from the right end $\bar{x}'$. Then the end slopes, found as end shears in the conjugate beam, are seen to be:

At the left end:

$$\theta_A = \frac{F\bar{x}'}{EIL} \tag{25}$$

At the right end:

$$\theta_B = -\frac{F\bar{x}}{EIL} \tag{26}$$

11. *Case in which the Cross Section of the Beam is not Constant.*—A single example will be sufficient in illustrating the use of the conjugate beam method in the

case of a varying cross section. We shall take the case of a simple beam carrying a concentrated load P at the center,—see Fig. 16. The moment of inertia is assumed to be I_0 on the middle half of the beam, $I_0/2$ on the outer quarters. The moment diagram is a triangle with altitude $PL/4$. Dividing the ordinates in the triangular moment diagram by EI_0 within the middle half of the span, and by $EI_0/2$ outside the middle half, we obtain the M/EI-diagram shown below in Fig. 16. This M/EI-load can be separated into the four triangles $I, II, III,$ and IV. Combining the reactions and bending moments produced by the triangular loads I and II on the one hand, III and IV on the other hand, we find the following expressions for the end shear and central moment in the conjugate beam; that is, for the end slope and central deflection of the given beam:

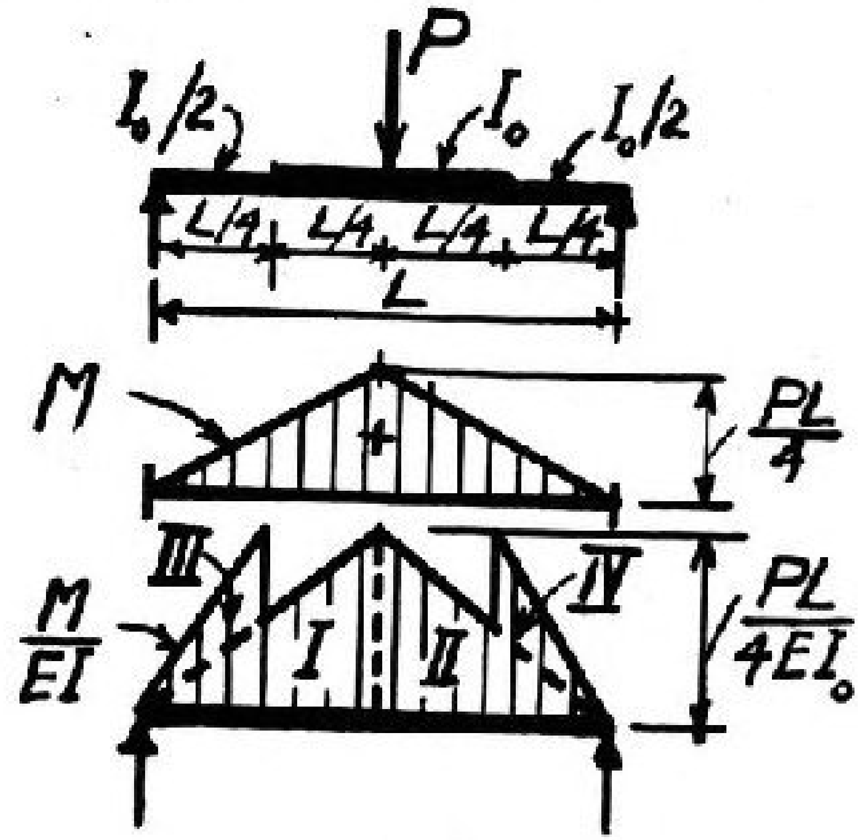

Fig. 16.

$$\theta_{max} = \frac{5\,PL^2}{64\,EI_0} \qquad y_{max} = \frac{3\,PL^3}{128\,EI_0} \tag{27}$$

12. *Beam with Overhanging Ends.*—The conjugate beam corresponding to a beam with overhanging ends was indicated in Fig. 3c. As an example illustrating the application of the conjugate beam method to beams of this and similar types we shall take the case shown in Fig. 17. The load consists of two equal forces P at the free ends. The overhanging ends have the same length a. The conjugate beam is indicated below. The M/EI-diagram is a trapezoid with altitude Pa/EI. This diagram, acting as a load on the conjugate beam, causes at the center of the middle span the following moment, which is equal to the deflection of the given beam at the center:

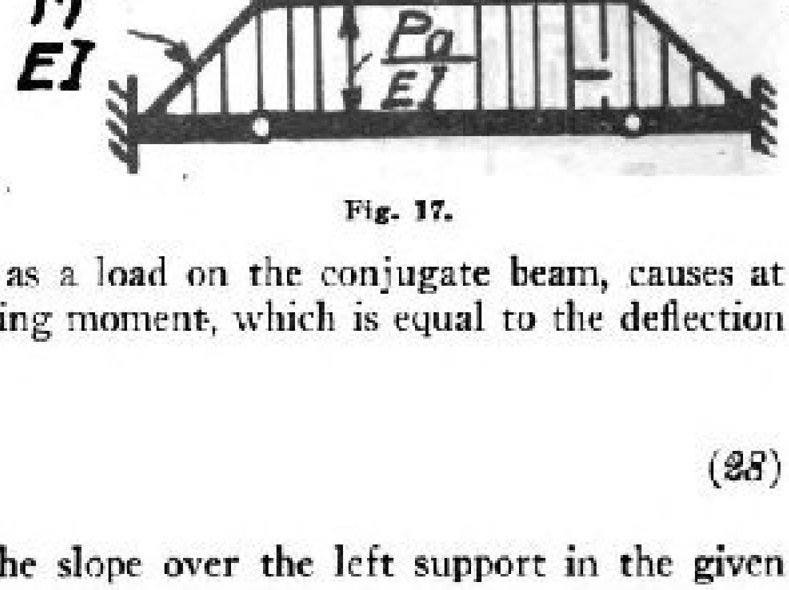

Fig. 17.

$$y_C = -\frac{PaL^2}{8EI} \tag{28}$$

At the left hinge the shear, equal to the slope over the left support in the given beam, is:

$$\theta_B = -\frac{PaL}{2EI} \tag{29}$$

At the left end the shear in the conjugate beam, or the slope of the given beam, is one-half the M/EI-area, that is,

$$\theta_A = -\frac{Pa(L+a)}{2EI} \tag{30}$$

The deflection at the left end, found as the moment in the conjugate beam, is:

$$y_A = \frac{Pa^2 L}{2EI} + \frac{Pa^3}{3EI} \tag{31}$$

III. STATICALLY INDETERMINATE BEAMS.

13. *General Remarks.*—The following cases of statically indeterminate beams were shown in Fig. 4: a beam fixed at both ends; a beam fixed at one end, simply supported at the other; and a continuous beam. / The general definition is recalled: beams are called statically indeterminate when, with given loads, it is not possible to determine the reactions, shears, and bending moments, without taking the deflections into consideration.

The problem involved may be separated into three parts. The first is to ascertain the character of the reactions, and of the shear and bending moment diagrams. The second is to determine the reactions and the moment diagrams quantitatively. The third is to find the slopes and the deflections. The first part is solved by comparing the given statically indeterminate beam with a "substitute beam" which is statically determinate, but which is made to deflect in the same way as the given beam by introducing certain supplementary loads at special points. The second part is solved conveniently by the conjugate beam method, by using such conjugate beams as were indicated in Fig. 4. Dimensions of the moment diagram are then determined by the condition that the conjugate beam must be in equilibrium. The third part can be solved by the conjugate beam method; when once the moment diagram of the given beam is known, then the slopes and deflections may be determined as shears and bending moments in the conjugate beam, just as when the given beam is statically determinate. Thus the conjugate beam method will serve the double purpose of determining not only the deflections and slopes, but also the moments, shears, and reactions.

The stiffness factor EI will again be assumed constant throughout the length of the beam, except when otherwise stated.

14. *Beams Fixed at Both Ends.*—Fig. 18*a* shows a single-spanned beam, fixed at both ends, and loaded in some general way. At each end there is a reaction consisting of a force which is numerically equal to the end shear, plus a couple, which is numerically equal to the bending moment at the end. The first step, according to the plan just outlined is to indicate the "substitute beam." The simple beam shown in Fig. 18*b* will be used. It has the same dimensions as the given beam. It carries the same loads throughout the length, except that at the ends the two couples $-M_A$ and $-M$ are applied as external loads, as are indicated in Fig. 18*b*. If it were not for the effect of these end couples, the deflections caused by downward loads would be decidedly greater in the substitute beam than in the given beam. We shall make the couples M_A and M_B equal to the bending moments at the ends of the given beam. The result is that when the reactions are considered as included in the sets of acting forces and couples, then the complete sets of forces and couples acting on the two beams are identical. It follows that the two beams

will have the same moment diagrams, and we may study the bending moments of the given beam by studying those of the substitute beam.

Fig. 18*c* shows what we shall call the "moment diagram of the simple span," that is, the moment diagram caused in the substitute beam by the given loads alone, when the end couples $-M_A$ and $-M_B$ are not acting. When the loads are all downward as in the figure, then these moments are all positive. In that case the end couples will have the directions indicated by the arrows in Fig. 18*b*. Such end

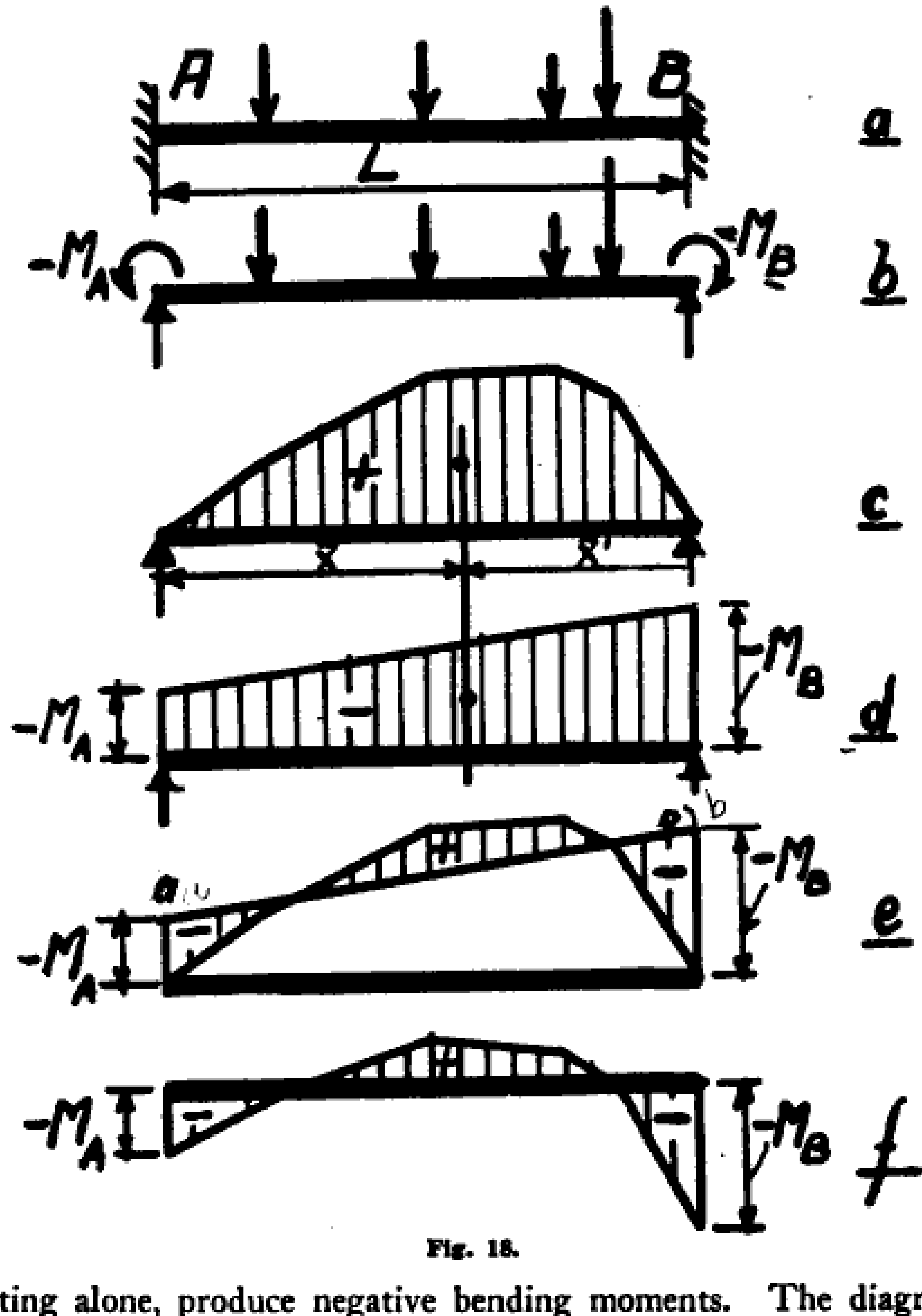

Fig. 18.

couples, acting alone, produce negative bending moments. The diagram of these moments is the trapezoid in Fig. 18*d*. By superimposing the "end moment trapezoid" *d* on the "simple span moment diagram" *c* the resultant moment diagram which is shown cross-hatched in Fig. 18*e* is obtained. In Fig. 18*f* the same diagram is shown, referred to a horizontal base.

We have now ascertained the character of the moment diagram, and we are ready for the second step in the analysis: to determine the yet unknown dimensions in the moment diagram, by using the conjugate beam method. That is, we shall determine the end moments M_A and M_B, or, in terms of graphics, complete the diagram in Fig. 18*e* by drawing the "closing line" *ab* in its correct position. The conjugate beam corresponding to a fixed-ended beam was indicated in Fig. 4*a*. It is an unsupported beam. Its load diagram is found by dividing the ordinates of the moment diagrams in Fig. 18*e* or *f* by the constant factor EI. This M/EI-diagram, acting as a load, must hold the unsupported conjugate beam in equilibrium. Or, the M/EI-diagram must hold itself in equilibrium. As the M/EI-diagram and the moment diagram are similar, EI being constant, it follows that also the moment diagram, considered as acting as a load on a free beam, must hold itself in equilibrium. Or, the diagrams in Fig. 18*c* and *d* must hold one another in equilibrium. This condition is brought about when minus the area in Fig. 20*d* is equal to the area in Fig. 20*c*, and when in addition the two areas have their centroids on the same vertical line. *Or, the end moment trapezoid has an area equal to but opposite the moment area of the simple span, and the two areas have their centroids on the same vertical line*.*

The following notation is used:

F = moment area of the simple span, that is area of the moment diagram which the given load would produce in the simple beam having the same span. (Fig. 18*c*).

$\bar{x}$ = horizontal distance of the centroid of the area F from the left end.

$\bar{x}'$ = horizontal distance of the centroid of F from the right end.

M_A = bending moment at the left end.

M_B = bending moment at the right end.

With this notation the condition that the two moment areas must be equal but opposite is expressed:

$$-(L/2)\ (M_A+M_B) = F \tag{32}$$

The condition that the two areas must have their centroids on a common vertical leads to the result that the moment of the trapezoid in Fig. 18*d* about the right end must be equal and opposite to the moment of the moment area of the simple span about the same point, or

$$-(L^2/6)\ (2M_A+M_B) = F\bar{x}' \tag{33}$$

Taking moments about the left end we find in the same way

$$-(L^2/6)\ (M_A+2M_B) = F\bar{x} \tag{34}$$

Any two of the equations (32), (33), and (34) determine the end moments M_A and M_B.

When the load is symmetrical, then the end-moment trapezoid becomes a rectangle, and each end moment becomes equal to minus the average ordinate of the simple span moment area, or

$$M_A = M_B = -F/L \tag{35}$$

Some of the important special cases will now be analyzed.

A. *Uniformly Distributed Load.* (Fig. 19).—The simple-span moment diagram is a parabola with maximum ordinate $M_0 = (1/8)wL^2$, and with total area $(2/3)\ M_0L$. Hence, by (35)

$$M_A = M_B = -(2/3)M_0 = -(1/12)wL^2 \tag{36}$$

November, 1921

*This theorem is stated by Maurice Levy in his Statique in Graphique. v. 2, 1886, p. 109.

The positive moment at the center is then

$$Mc = (1/3)M_o = (1/24)wL^2 \tag{37}$$

It should be noted that these formulas apply only when the ends are absolutely rigidly fixed. Actual end conditions rarely furnish more than partial rigidity; hence the coefficient $1/24$ in (37) must be used only with great caution, and in most practical problems it should be replaced by some higher value, such as $1/12$.

We shall use the case to illustrate how the third step in the complete analysis may pe performed: namely, the determination of the deflections. Two solutions will be indicated, in both of which the conjugate beam method will be used. In the first solution the deflections of the given beam are found directly as bending moments in the unsupported conjugate beam. The M/EI-load is shown below in Fig. 19.

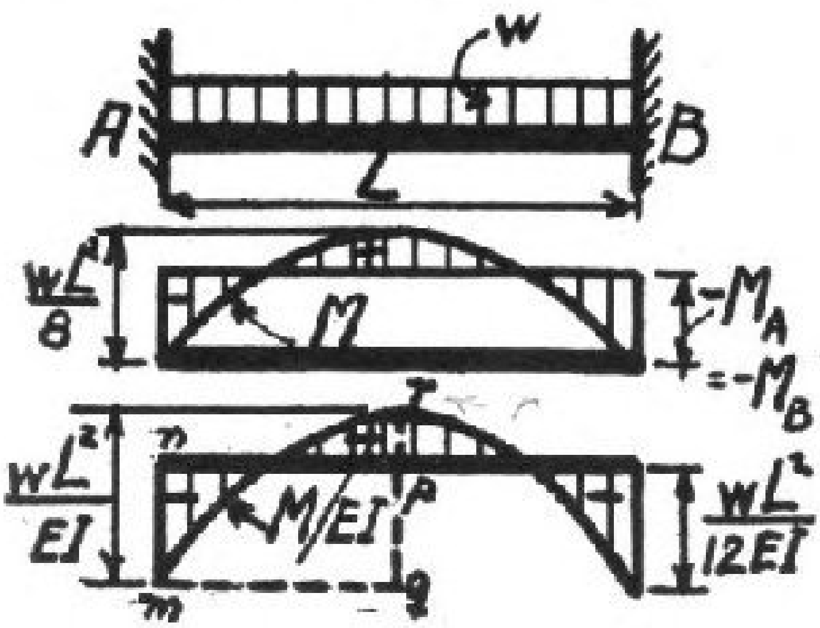

Fig. 19.

Assume that the object is to find the maximum deflection of the given beam. It occurs at the center and is equal to the bending moment in the conjugate beam at that point. The M/EI-load to the left of the center may be considered as consisting of two parts: the rectangle $mnpq$ acting as an upward load, combined with the parabolic area mrq acting as a downward load. These two areas have the same size, $wL^3/(24EI)$. Their centroids are at the horizontal distances $L/4$ and $5L/16$, respectively, from the left end; that is, their mutual horizontal distance is $L/16$. The couple formed by these two loads is equal to the moment in the conjugate beam at the center, that is, equal to the deflection y_{max} of the given beam at the center. This gives the computation:

$$y_{max} = \frac{wL^3}{24EI} \cdot \frac{L}{16} = \frac{wL^4}{384EI} \tag{38}$$

This deflection is seen to be only one-fifth of that found for the corresponding simple beam,—see formula (17).

The other method of determining the deflections is based on the principle that the substitute beam (see Fig. 18*b*) is made to deflect in the same way as the given beam. Since the substitute beam is a simple beam, its conjugate beam is a simple beam. Thus we may use a simple beam as a "substitute conjugate beam." This procedure has the advantage that it allows us to separate the M/EI-load into two parts, each of which, acting alone, would not hold the unsupported conjugate beam in equilibrium. As the one part we take the parabolic M/EI-area corresponding to the simple span, as the other the rectangle corresponding to the end moments alone. The moment at the center due to the simple-span parabolic area was found previously. It is the simple-span deflection derived in formula (17), equal to $(5/384EI)wL^4$. The rectangular diagram caused by the end moments has an altitude $(-1/12EI)wL^2$. It represents a uniformly distributed load, and gives therefore a moment at the center equal to $-(1/8)\,(1/12EI)wL^4$. Superposition of the two moments gives the following computation of the resultant moment at the center, equal to the deflection of the given beam,

$$y_{max} = \left(\frac{5}{384} - \frac{1}{8} \cdot \frac{1}{12}\right)\frac{wL^4}{EI} = \frac{wL^4}{384EI}$$

which is the result expressed in formula (38).

B. *Concentrated Load at Center* (Fig. 20.).—The simple-span moment diagram is a triangle, with altitude $(1/4)PL$. The average ordinate is one-half the altitude; hence the end moments are

$$M_A = M_B = -(1/8)PL, \quad (39)$$

and the moment at the center is

$$M_C = +(1/8)PL \quad (40)$$

In the same way as in the preceding case, by the use, for example, of the second scheme of computation, the central deflection is expressed:

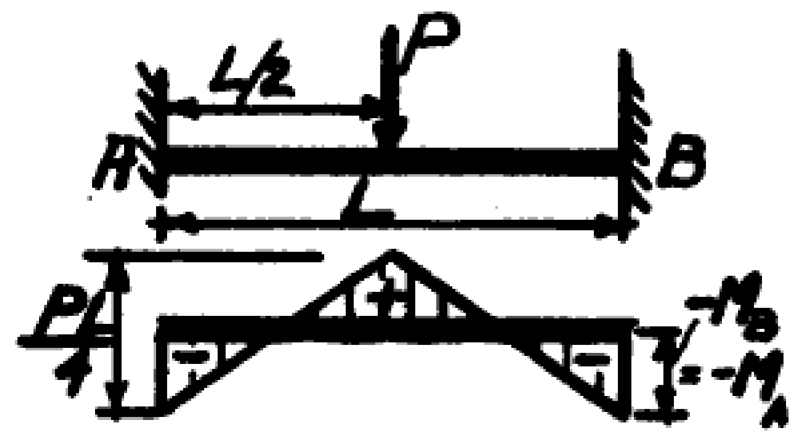

Fig. 20.

$$y_{max} = \left(\frac{1}{48} - \frac{1}{8} \cdot \frac{1}{8}\right)\frac{PL^3}{EI} = \frac{PL^3}{192EI} \quad (41)$$

C. *Concentrated Load at Any Point* (Fig. 21).—The load P is at the distances a from the left end, b from the right end. The simple-span moment diagram is a triangle with altitude Pab/L, area $F = Pab/2$, and with the center of gravity distances $\bar{x} = (L+a)/3$ from the left end, and $\bar{x}' = (L+b)/3$ from the right end. By substituting these values in the general equations (33) and (34) these equations become

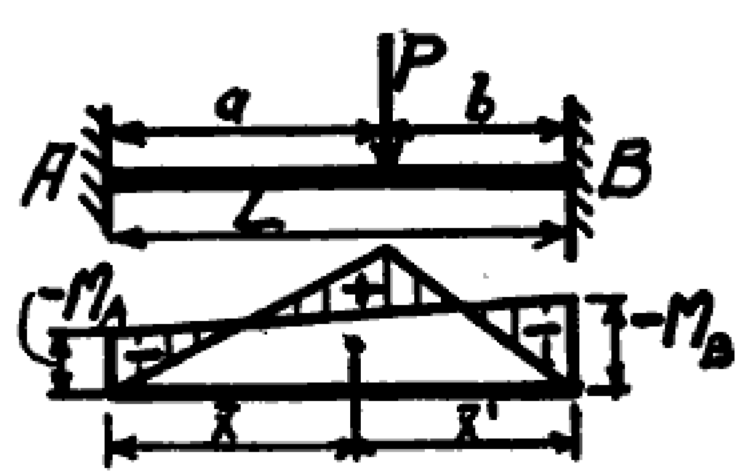

Fig. 21.

$$-(L^2/6)\ (2M_A + M_B) = (Pab/2) \cdot (L+b)/3$$

$$-(L^2/6)\ (M_A + 2M_B) = (Pab/2) \cdot (L+a)/3$$

Solving these equations we find

$$M_A = -Pab^2/L^2 \quad ; \quad M_B = -Pa^2b/L^2 \quad (42)$$

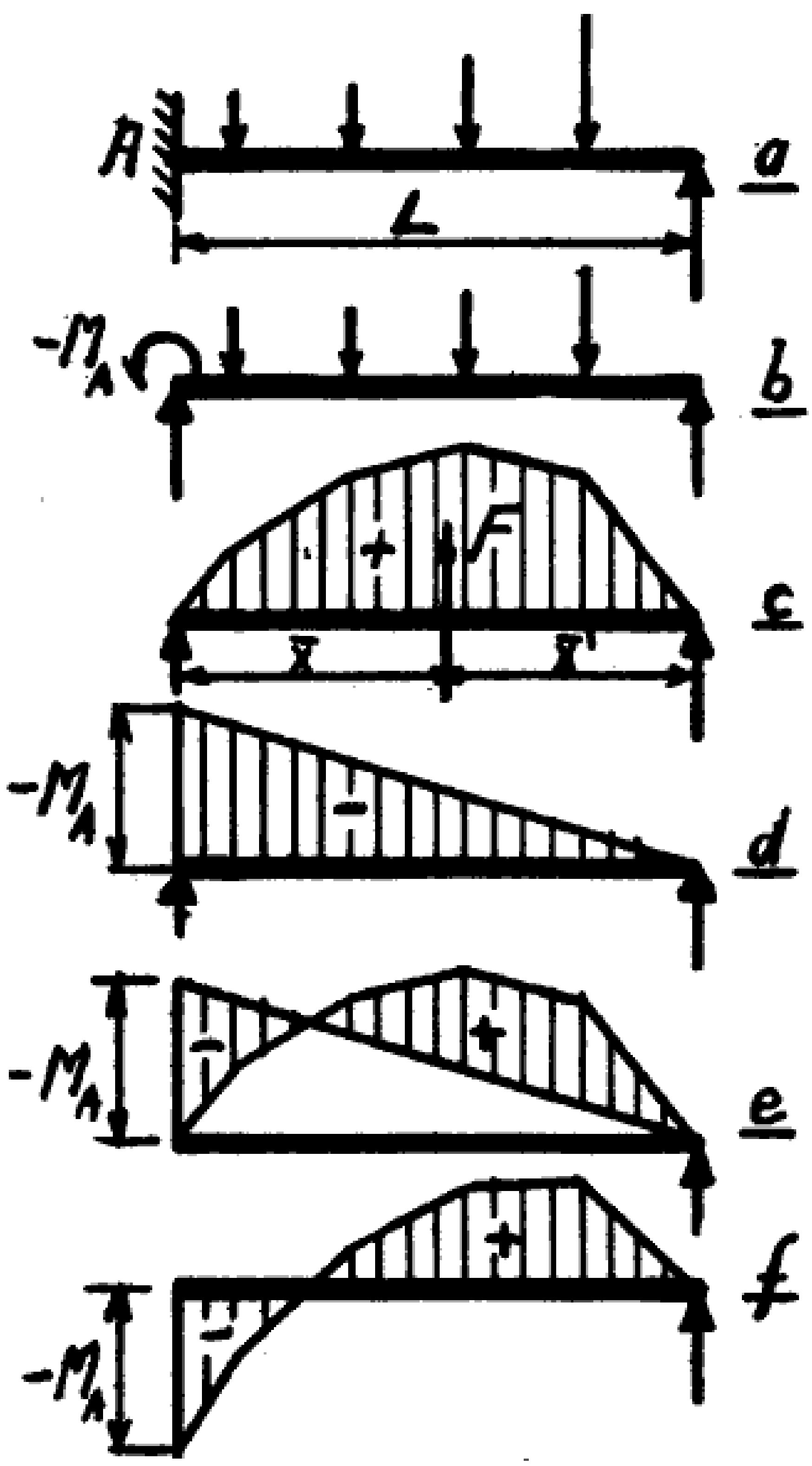

Fig. 22.

15. *Beams Fixed at One End, Simply Supported at the Other End.*—The beam in Fig. 22*a* is fixed at the left end *A*, simply supported at the right end. Its load is of a general type. In analyzing this case a procedure may be followed which is quite similar to that of the preceding case. The simple beam in Fig. 22*b* is introduced as a substitute beam. The load applied is that of the given beam plus an end couple $-M_A$ acting as shown and equal to the couple-component of the reaction of the given beam at *A*. The result is that the two beams will act alike. The moment diagram of either beam can therefore be found by superposition of the following two component parts, derived by consideration of the substitute beam: the simple-span moment area shown in Fig. 22*c*; and the end moment diagram, here a triangle, shown in Fig. 22*d*. The resultant diagram is shown cross-hatched in Fig. 22*e*. The resultant moments are measured there from the inclined closing line. The same diagram, referred to a horizontal base, is shown in Fig. 22*f*.

A conjugate beam representing this case was indicated in Fig. 4*b*: it is free at the left end, simply supported at the right end. It is incompletely supported, but it will be in equilibrium if the moment of the M/EI-diagram about the right end is zero. As EI is constant, this condition will be satisfied when the moment diagram itself considered as a load has a zero moment about the right end. We denote again: F=simple span moment area $\bar{x}$ and $\bar{x}'$= center of gravity distances (see Fig. 22*c*). By combining the moments of the two component parts of the moment area (Fig. 22 *c* and *d*) the condition is found

$$F\bar{x}' - \frac{L(-M_A)}{2} \cdot \frac{2L}{3} = 0$$

or

$$M_A = -3F\bar{x}'/L^2 \qquad (43)$$

This formula is the same as (33) with $M_B = O$.

Two definite cases will now be analyzed.

A. *Uniformly Distributed Load w.* (Fig. 23).—The maximum simple-span moment is $M_0 = (1/8)wL^2$. Substituting in (43) $F = (2/3)ML_0$ $\overline{x'} = L/2$ we find for the end moment

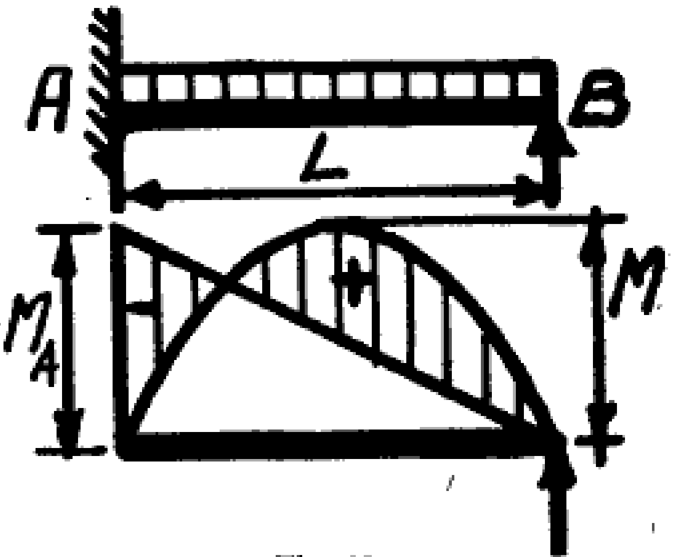

Fig. 23.

$$M_A = -M_0 = -(1/8)wL^2 \tag{44}$$

The reaction, for example, at the fixed end, may be found as follows: It is equal to the end shear, which is equal to the end slope in the resultant moment diagram. This slope is found by combining the slope of the closing line with the end slope in the simple-span moment diagram. The latter is the same as the end shear in the simple span, or $wL/2$. The slope of the closing line is numerically $wL/8$. A consideration of the diagram in Fig. 23 shows that the slopes must be combined by adding their numerical values. Hence the values of the two reactions are

$$A = (5/8)wL \quad , \quad B = (3/8)wL \tag{45}$$

It follows that the point of the given beam where the shear is zero, is at the distance $(3/8)L$ from the right end. The maximum positive moment occurs at that point and is found to be

$$M_{max} = +(9/128)wL^2 \tag{46}$$

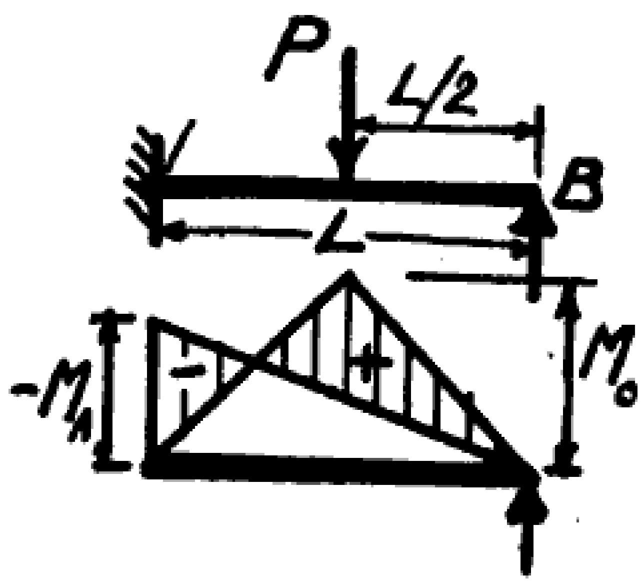

Fig. 24.

B. *Concentrated Load at Center.* (Fig. 24).—The maximum simple-span moment is $M_0 = (1/4)PL$. Substituting in (43) $F = (1/2)M_0L$, $\overline{x'} = L/2$ we find the end moment

$$M_A = -(3/4)M_0 = -(3/16)PL \tag{47}$$

Then, from the shape of the diagram the maximum positive moment, occurring under the load, is found to be

$$M_P = (5/8)M_0 = (5/32)PL \tag{48}$$

As in the preceding case the reactions may be found by adding $\pm M_A/L$ to the simple span reactions. The values are then found:

$$A = (11/16)P \quad , \quad B = (5/16)P \tag{49}$$

16. *Continuous Beams with Constant Cross Section and with Supports on the Same Level.*—Fig. 25*a* shows two consecutive spans of a continuous beam loaded in some general way. EI is constant. The supports are simple supports. The supports and spans are numbered from the left end, and in that way the two spans shown are the *nth* and $(n+1)th$. The beam system in Fig. 25*b* is introduced as "substitute beam." It differs from the given beam by having hinges over the supports but otherwise it has the same dimensions as the given beam. It carries the same loads as the given beam, but in addition, each span is loaded by end couples, such as the couples $-M_{n-1}$, $-M_n$, and $-M_{n+1}$ which are indicated in Fig. 25*b*. By choosing M_{n-1}, M_n, and M_{n+1} equal to the bending moments in the given beam over the corresponding supports the substitute beam is made to act like the given beam, and the two beams will then have identical deflections, bending moments, etc.

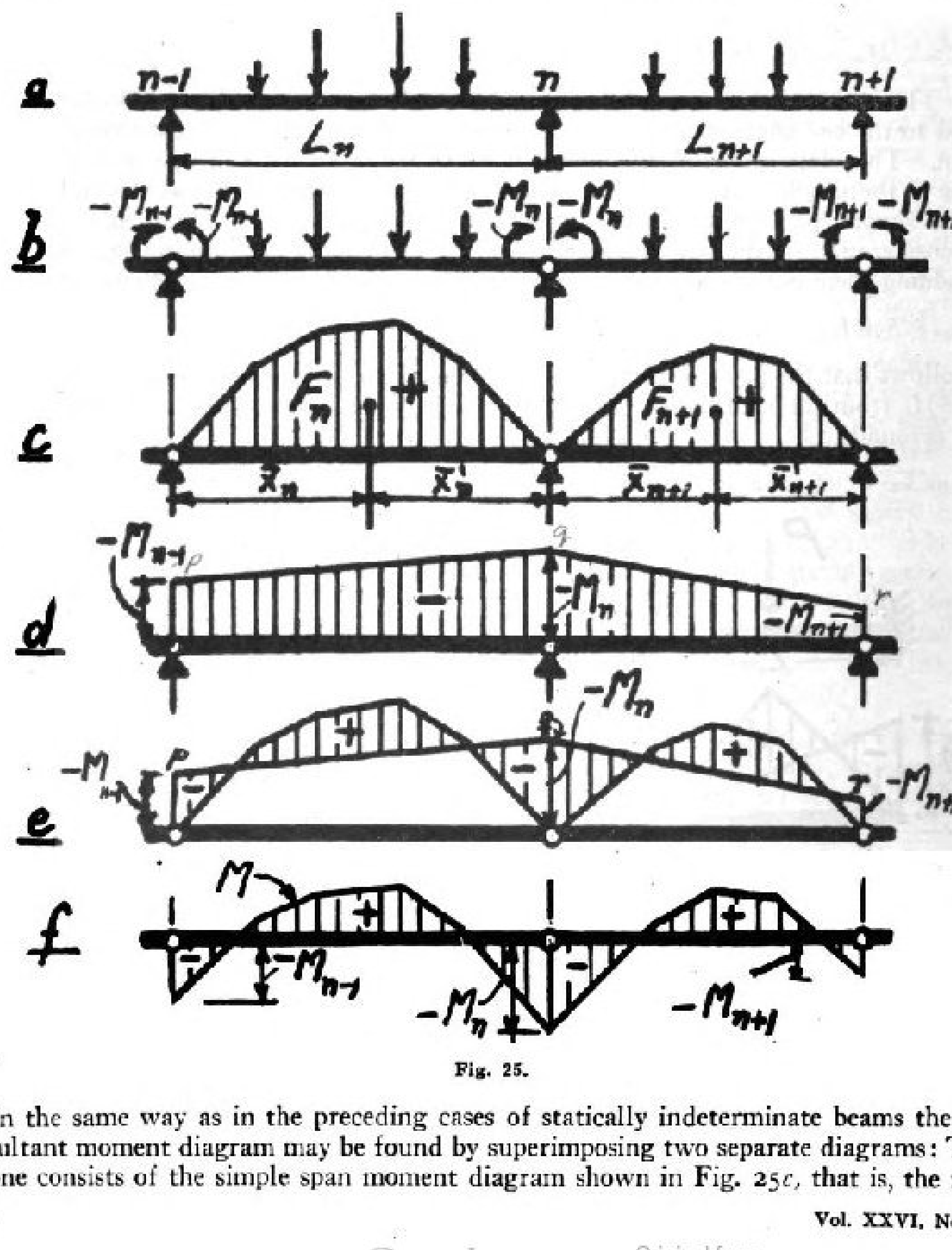

Fig. 25.

In the same way as in the preceding cases of statically indeterminate beams the resultant moment diagram may be found by superimposing two separate diagrams: The one consists of the simple span moment diagram shown in Fig. 25*c*, that is, the moment diagrams in the substitute beam caused by the original given loads only, without the influence of the end couples. The other is the diagram in Fig. 25*d*, which is produced in the substitute beam by the end moments alone. It consists of a series of trapezoids, of which the upper sides are connected into a polygon. The resultant diagram is the cross-hatched area in Fig. 25*e*. Fig. 25*f* shows this diagram referred to a horizontal base. In accordance with the usage which was introduced in the preceding cases of statically indeterminate beams, the polygon *pqr* in Fig. 25*d* is called the closing line.

The nature of the conjugate beam was indicated in Fig. 4*c*. At the supports over which there is continuity the conjugate beam has an unsupported hinge. This conjugate beam is "incompletely supported." But it will be in equilibrium provided the M/EI-load satisfies certain special conditions, which may be stated as follows: We replace the conjugate beam temporarily by a "substitute conjugate beam" of the folowing description: it carries the same M/EI-load as the original conjugate beam, and it has the same hinges; the only difference is that simple supports are brought up under the hinges which were unsupported in the original conjugate beam. Assume now that the M/EI-load causes the reactions from all such supports to be zero. In that case the substitute conjugate beam, with supported hinges, and the original conjugate beam, with unsupported hinges, will act alike. That is, an M/EI-load adjusted in such a way will hold the original conjugate beam in equilibrium. The substitute conjugate beam is in fact a series of simple beams. It may be interpreted as the conjugate beam corresponding to the substitute beam in Fig. 25*b*.

EI was assumed constant. Hence, without disturbing the equilibrium, the M/EI-diagram can be replaced by the moment diagram itself, acting as a load. This consideration leads to the following law, which is the equivalent of the original conditions: the moment diagram acting as a load on the substitute conjugate beam must cause the reactions of its intermediate supports to be zero. This law will now be expressed in terms of equations. The notation is:

L_n = length of the *nth* span.
F_n = area of the simple-span moment diagram in the *nth* span (Fig. 27*c*).
$\bar{x}_n$ = horizontal distance of the centroid of F_n from the left end of the *nth* span.
$\bar{x}'_n$ = horizontal distance of same point from the right end of the span.
M_n = moment in given beam over the *nth* support.
Indices $n+1$ and $n-1$ are used in the same way as n.

We shall especially consider the middle support in Fig. 25*e*. Its reaction may be expressed as the sum of the reactions due to the separate loads in Fig. 25*d* and *c*. The diagram in Fig. 25*d*, acting alone as a load, would give the reaction

$$(L_n/6)\,(M_{n-1}+2M_n) + (L_{n+1}/6)\,(2M_n+M_{n+1}) \tag{50}$$

In a similar way the reaction due to the moment diagram of the simple span in Fig. 25*c* is found to be

$$F_n\bar{x}_n/L_n + F_{n+1}\,\bar{x}'_{n+1}/L_{n+1} \tag{51}$$

By putting the one of these reactions equal to but opposite the other the following equation is found:

$$-M_{n-1}L_n - 2M_n(L_n+L_{n+1}) - M_{n+1}L_{n+1} = 6F_n\bar{x}_n/L_n + 6F_{n+1}\bar{x}'_{n+1}/L_{n+1} \tag{52}$$

This equation gives a relation between the moments over three consecutive supports. It is known as the *theorem of three moments*. By writing one such equation for each support over which the beam is continuous, a number of equations is found, equal to the number of unknown moments over such supports. The theorem of three moments represents, therefore, a solution of the problem of continuous beams over simple supports.

The reaction of the given beam, for example, at the middle support in Fig. 25, may be found as the increase of shear when passing from the left to the right of the support. This increase is the same as the increase of slope in the moment diagram *f* when passing the same point. It may be found as the sum of the corresponding increases in the diagrams *c* and *d*. The increase of slope at the middle support in *c* is the same as the simple-span reaction, that is, the reaction produced in the substitute beam in Fig. 25*b* when the given load acts alone without the end couples shown in that figure. To this reaction we add the increase of slope at the middle support in the diagram *d*. Since the ordinates of the diagram in Fig. 25*d* are negative, this increase is positive in the particular case in Fig. 25, and the effect of the continuity is an increase of the middle reaction. In any case this increase may be expressed as

$$\frac{M_{n+1}-M_n}{L_{n+1}} - \frac{M_n-M_{n-1}}{L_n} \qquad (53)$$

Some particular cases will now be treated.

A. *Uniformly Distributed Loads.*—We assume the following uniformly distributed loads:

w_n = load per unit length in span *n*, constant throughout the length of the span,

w_{n+1} = same in span $n+1$.

In that case we have: $F_n = (2/3)\ (1/8)\ wL^3 = (1/12)wL^3$ and $\bar{x}_n = L_n/2$, and similar expressions for span $n+1$. By substitution of these values the equation of three moments, (52), becomes

$$-M_{n-1}L_n - 2M_n(L_n+L_{n+1}) - M_{n+1}L_{n+1} = (1/4)w_nL_n^3 + (1/4)w_{n+1}L^3_{n+1} \qquad (54)$$

When the spans are equal, $L_n = L_{n+1} = L$, this equation becomes

$$-M_{n+1} - 4M_n - M_{n+1} = (1/4)\ (w_n+w_{n+1})L^2 \qquad (55)$$

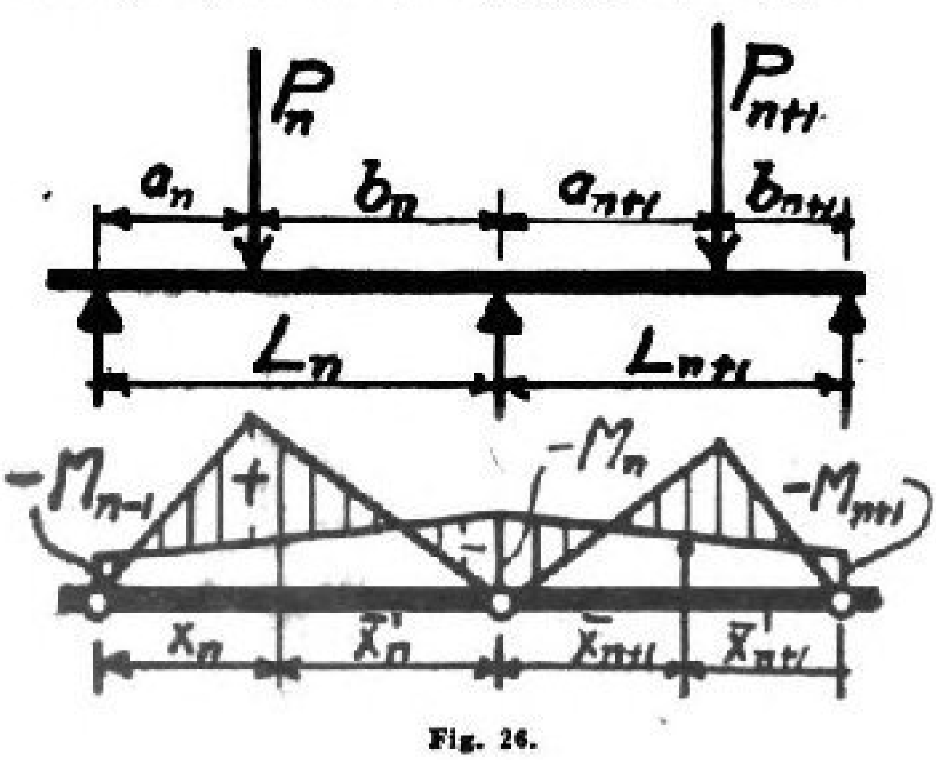

Fig. 26.

The equation (54) was derived by *Bertot* in 1855 and was also indicated by *Clapeyron* in 1857.* The general form (52) was derived by *Bresse*.** The three moment equation, whether in the general or special form, is often called *Clapeyron's equation*.

B. *One Concentrated Force in Each Span.*—Fig. 26 illustrates the case and gives the notation. The simple span moment area for each span is a triangle, with dimensions defined as in Fig. 13. The following values are found:

$$F_n = (1/2)P_n a_n b_n \quad ; \quad \bar{x}_n = (1/3)(L_n + a_n) \quad ;$$
$$F_{n+1} = (1/2)P_{n+1} a_{n+1} b_{n+1} \quad ; \quad \bar{x}'_{n+1} = (1/3)(L_{n+1} + b_{n+1})$$

Substitution of these values in the general equation (52) gives the three moment equation

$$-M_{n-1}L_n - 2M_n(L_n + L_{n+1}) - M_{n+1}L_{n+1} = \frac{P_n a_n b_n (L_n + a_n)}{L_n} + \frac{P_{n+1} a_{n+1} b_{n+1} (L_{n+1} + b_{n+1})}{L_{n+1}} \tag{56}$$

C. *Each Span Carries a Uniform Load and, in Addition, Several Concentrated Forces.*—This case is solved by superposition of solutions of equations (54) and (56). By adding the right side expressions the following resultant three-moment equation is found

$$-M_{n-1}L_n - 2M_n\,(L_n + L_{n+1}) - M_{n+1}\,L_{n+1} = \frac{w_n L_n^3}{4} + \frac{w_{n+1}L^3_{n+1}}{4} + \frac{\Sigma P_n a_n b_n\ (L_n + a_n)}{L_n} + \frac{\Sigma\, P_{n+1} a_{n+1} b_{n+1} (L_{n+1} + b_{n+1})}{L_{n+1}} \tag{57}$$

This three-moment equation is of a rather general form. The summations on the right side may easily be replaced by integrals.

17. *Continuous Beams Having Different Cross-Sections in Different Spans, But With the Supports at the Same Level.*—It is assumed that the cross-section is constant throughout each span, but that it may vary from one span to another. We denote:

I_n = moment of inertia of cross-section in *nth* span.
I_{n+1} = same in $(n+1)$*th* span.

The same procedure may be followed as when the cross-section is constant, except that in the present case the M/EI-load can not be replaced by the moment diagram itself as a load on the conjugate beam. A solution is found by transforming the expressions (50) and (51), which represent the reactions due to M-loads, as follows: the parts of these reactions caused by the loads on the *nth* span are divided by EI_n, while the parts caused by the loads on the $(n+1)$*th* span are divided by EI_{n+1}. Thereby (50) and (51) are changed into the following expressions representing reactions at the middle support in Fig. 25*e*:

$$(L_n/6EI_n)\ (M_{n-1} + 2M_n) + (L_{n+1}/6EI_{n+1})\ (2M_n + M_{n+1}) \tag{58}$$

and

$$F_n\bar{x}_n/EI_nL_n + F_{n+1}\bar{x}'_{n+1}/EI_{n+1}L_{n+1} \tag{59}$$

Expressing these two reactions as equal and opposite we find the three-moment equation

$$-M_{n-1}L_n/I_n - 2M_n\ (L_n/I_n + L_{n+1}/I_{n+1}) - M_{n+1}L_{n+1}/I_{n+1} = \frac{6F_n\bar{x}_n}{I_nL_n} + \frac{6F_{n+1}\bar{x}'_{n+1}}{I_{n+1}L_{n+1}} \tag{60}$$

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*Bertot, Comptes Rendus de la Société des Ingénieurs Civils, 1855, p. 278. Clapeyron, Comptes Rendus, v. 45, 1857, p. 1076.
**According to A. Ostenfeld, Teknisk Elasticitetslære, 3 ed. 1916, p. 216.

Equation (60) includes (52) as a special case: namely, that in which $I_n = I_{n+1}$. When the load on the two spans consists of the uniformly distributed loads w_n and w_{n+1}, then (60) takes the following special form, which corresponds to (54), and includes (54) as a special case:

$$-M_{n-1}L_n/I_n - 2M_n\,(L_n/I_n + L_{n+1}/I_{n+1}) - M_{n+1}L_{n+1}/I_{n+1} = \frac{w_nL^3_n}{4I_n} + \frac{w_{n+1}L^3_{n+1}}{4I_{n+1}} \qquad (61)$$

18. *Continuous Beams with Supports Out of Level.*—The case is represented in Fig. 27*a* where three consecutive supports are shown with the deflections y_{n-1}, y_n, and y_{n+1}. Such movements of the supports out of level would cause stresses in the continuous beam even if no vertical loads were carried. We shall assume that the deflections y_{n-1}, y_n, and y_{n+1} are definite quantities which are known beforehand.

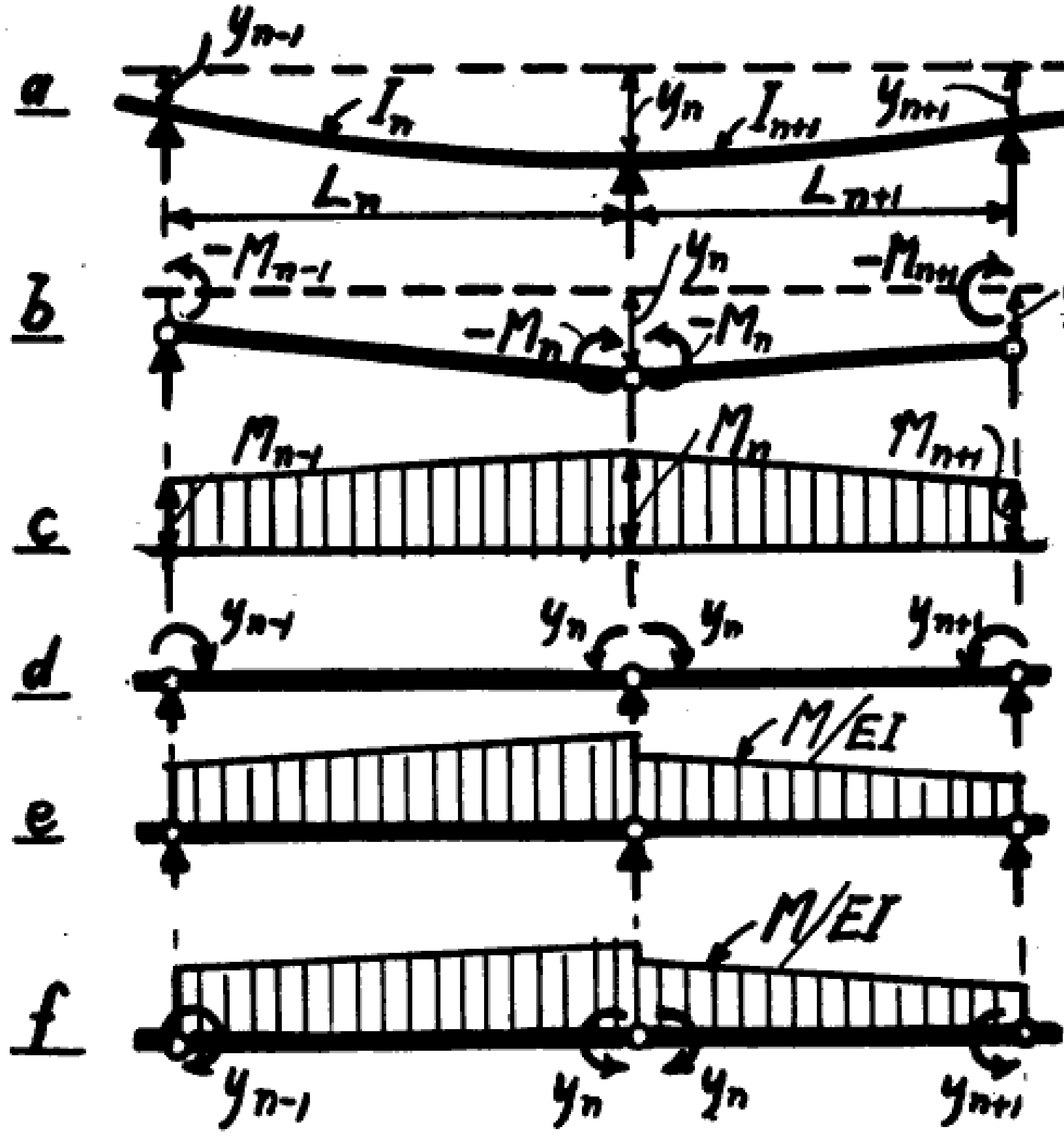

Fig. 27.

Fig. 27*b* shows a substitute beam which has the following properties: its supports are displaced as those of the given beam; it has hinges over the supports; each span is loaded by end couples, such as $-M_{n-1}$, $-M_n$, etc., where M_n= bending moment in the original beam over the *nth* support; in other respects the two beams are alike. Under these circumstances the beams *a* and *b* will deflect alike, and the bending moments in both beams will be of the types shown in Fig. 27*c*.

The conjugate beam with its loads is shown in Fig. 27*f*. The M/EI-load corresponds to the moment diagram in Fig. 27*c*. The conjugate beam has unsupported hinges. Right at these hinges there must be bending moments equal to the deflections y_{n-1}, y_n, and y_{n+1} in Fig. 27*a*. These bending moments must therefore be transferred to the beam by pairs of external couples acting at the hinges as indicated by the arrows in Fig. 27*f*. These loads must hold the conjugate beam in equilibrium. Fig. 27*d* and *e* show the end couples and the distributed M/EI-load separately, acting on a substitute beam which has simply supported hinges. The middle reaction in *d* is

$$\frac{y_{n-1}-y_n}{L_n}+\frac{y_{n+1}-y_n}{L_{n+1}} \tag{62}$$

The reaction in Fig. 27*e* is the same as that already indicated by expression (58). By expressing the reactions in *d* and *e* as equal but opposite we find the following three-moment equation, which represents the effect of the supports being out of level:

$$\begin{aligned} -M_{n-1}L_n/I_n-2M_n\,(L_n/I_n+L_{n+1}/I_{n+1})\,- \\ M_{n+1}L_{n+1}/I_{n+1}=6E\left(\frac{y_{n-1}-y_n}{L_n}+\frac{y_{n+1}-y_n}{L_{n+1}}\right) \end{aligned} \tag{63}$$

The right side in this equation may be simplified by a particular choice of the line from which the deflections are measured. In Fig. 28, AB is the original undeflected center line. But we shall now measure the deflections from the chord CD. Then if we write $y_{n-1} = y_{n+1} = 0$, $y_n = \delta_n$, the right side in (63) becomes

$$-6E\delta_n\,(1/L_n+1/L_{n+1}) \tag{64}$$

Substituting this expression in (63), and combining with the effect of vertical loads, as expressed by equation (60), we find the following general equation of three moments:

$$\begin{aligned} &-M_{n-1}L_n/I_n-2M_n\,(L_n/I_n+L_{n+1}/I_{n+1})-M_{n+1}L_{n+1}/I_{n+1} \\ &=\frac{6F_n\bar{x}_n}{I_nL_n}+\frac{6F_{n+1}\bar{x}'_{n+1}}{I_{n+1}L_{n+1}}-6E\delta_n\left(\frac{1}{L_n}+\frac{1}{L_{n+1}}\right) \end{aligned} \tag{65}$$

This equation takes into account both the ordinary bending effects, expressed by the moment areas F_n and F_{n+1}, and the out-of-level effect, defined by the distance δ_n in Fig. 28.

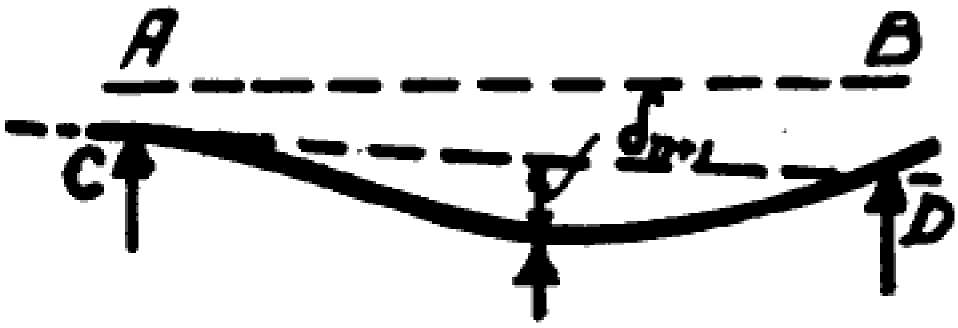

Fig. 28.

When the cross-section is constant throughout the length of the two spans, as expressed by $I_n = I_{n+1} = I$, then equation (65) takes the form:

$$-M_{n-1}L_n - 2M_n\,(L_n + L_{n+1}) - M_{n+1}L_{n+1}$$
$$= \frac{6F_n\overline{x}_n}{L_n} + \frac{6F_{n+1}\overline{x}'_{n+1}}{L_{n+1}} - 6EI\delta_n\left(\frac{1}{L_n} + \frac{1}{L_{n+1}}\right) \qquad (66)$$

This is the general three-moment equation in case of constant cross-section.